The PRIMES 2022 problem set
General math problems

G1. Consider a plane passing through the midpoints of two opposite edges of a regular tetrahedron. The projection of the tetrahedron to this plane is a quadrilateral of area A with one of the angles 60°. Find the surface area of the tetrahedron.

Solution. Let A, B, C and D be the vertices of the tetrahedron and assume the plane P described in the problem intersects $\overline{AB}$ and $\overline{CD}$ at their midpoints M and N. Let A_1, B_1, C_1 and D_1 be the projections of A, B, C and D to P.

First, we observe that $A_1B_1C_1D_1$ is an isosceles trapezoid. In fact, since $\overline{NA} = \overline{NB}$ and $\angle ANA_1 = \angle BNB_1$, we know that $\overline{NA_1} = \overline{NB_1}$. In particular, this implies $\overline{NM} \perp \overline{A_1B_1}$ since M is also the midpoint of $\overline{A_1B_1}$. Similarly, we can see that $\overline{MN} \perp \overline{C_1D_1}$. As a result, we can conclude that $\overline{A_1D_1} = \overline{C_1B_1}$.

Next, we denote $\alpha := \angle AMA_1$ and $\ell := \overline{AB}$. The area of $A_1B_1C_1D_1$ is

$$\begin{aligned} A = \frac{\overline{MN}}{2}(\overline{A_1B_1} + \overline{C_1D_1}) &= \frac{\ell}{2\sqrt{2}}\left(\overline{AB}\cos\alpha + \overline{CD}\cos\left(\frac{\pi}{2} - \alpha\right)\right) \\ &= \frac{\ell^2}{2\sqrt{2}}(\cos\alpha + \sin\alpha), \end{aligned}$$

so the surface area of the tetrahedron is

$$\sqrt{3} \cdot \ell^2 = \frac{2\sqrt{6}}{\cos\alpha + \sin\alpha} A.$$

Finally, we find out the value of $\cos\alpha + \sin\alpha$. Since $\angle AMA_1 = \alpha$, we know $\angle DND_1 = \frac{\pi}{2} - \alpha$. Then, we have

$$\overline{A_1B_1} - \overline{C_1D_1} = \ell \cdot \cos\alpha - \ell \cdot \sin\alpha = \ell \cdot (\cos\alpha - \sin\alpha).$$

On the other hand, using the fact that one of the angles of $A_1B_1C_1D_1$ is $60^\circ = \frac{\pi}{3}$, WLOG say $\angle D_1A_1B_1 = \frac{\pi}{3}$, we can derive

$$\overline{A_1B_1} - \overline{C_1D_1} = \overline{A_1D_1} = \frac{2}{\sqrt{3}}\overline{MN} = \frac{2\ell}{\sqrt{6}}.$$

Therefore, we get

$$\cos\alpha - \sin\alpha = \frac{2}{\sqrt{6}},$$

which implies $2\cos\alpha \cdot \sin\alpha = \frac{1}{3}$ and hence $\cos\alpha + \sin\alpha = \frac{2}{\sqrt{3}}$. Consequently, we have that the surface area of the tetrahedron is

$$\sqrt{3} \cdot \ell^2 = \frac{2\sqrt{6}}{\cos\alpha + \sin\alpha} A = 3\sqrt{2}A.$$

G2. For an m-digit number A and $(n-m)$-digit number B let $A \circ B$ be the n-digit number obtained by concatenation of A and B (where we allow the leftmost digit to be zero). For example, if $m = 2, n = 5$, $A = 23$, $B = 045$, then $A \circ B = 23045$ and $B \circ A = 04523$.

From now on assume that $m = 2$. Let k be a 2-digit number, and consider the equation

$$\frac{B \circ A}{A \circ B} = k$$

with $A > 0$ and any $n \geq 3$. It is clear that if $X := A \circ B$ is a solution of this equation then so is $X \circ X$, $X \circ X \circ X$, etc. We say that a solution X is *primitive* if it is not obtained in this way, by concatenating a smaller solution with itself several times.

(a) Find all primitive solutions for $k = 9$ and $k = 15$.

(b) Describe all primitive solutions for general k. Are there finitely many?

Solution. Consider the decimal

$$x = 0.ABAB.... = \frac{A \circ B}{10^n - 1}.$$

Then $100x = A.BAB....$ So we get

$$100x - A = kx,$$

thus

$$x = \frac{A}{100 - k}.$$

So we get

$$\frac{A}{100 - k} = \frac{A \circ B}{10^n - 1}.$$

In particular,

$$\frac{A}{100 - k} < \frac{A + 1}{100},$$

i.e.,

$$k(A + 1) < 100.$$

Thus all solutions are obtained by running through pairs of 2-digit positive numbers k, A such that $k(A + 1) < 100$ and $d := \frac{100-k}{GCD(100-k,A)}$ is coprime to 10. For each such k, A, let r be the multiplicative order of 10 modulo d (i.e., in $(\mathbb{Z}/d\mathbb{Z})^\times$). Then $n = qr$, $q \geq 1$, so

$$A \circ B = \frac{A(10^{qr} - 1)}{100 - k}.$$

Primitive solutions correspond to $q = 1$, so there are finitely many. There is a unique primitive solution for each A, k, which is defined by the equality $\frac{A}{100-k} = 0.\overline{X}$.

In particular, for $k = 9$ we have $1 \le A \le 10$, so we have the primitive solutions $A \cdot Y$ where $Y = 010989$. For $k = 15$ we get $A = 5$ and only one primitive solution $X = 0588235294117647$.

G3. Let m be a fixed positive integer, and consider the following game. At each move, you pick uniformly at random an integer $0 \le k \le m$. Then you score k points, but only if k does not exceed the smallest previously picked number (otherwise you don't score any points on that move). For example, if $m = 3$ and your random numbers are $2, 3, 1, 2, 1, 0, 3, \ldots$ then you score only on the 1st, 3rd and 5th move and don't score anything after the 5th move, so you total score is $2 + 1 + 1 = 4$.

(i) How much will you score on average if you play indefinitely?

(ii) Let $a(n, m)$ be the average amount you score in n steps. Find a closed formula for $a(n, 1)$ and $a(n, 2)$.

(iii) Find a closed formula for $a(n, m)$.

Solution. Let $a(n) := a(n, m)$ and $a_r(n)$ be the average amount you score in n moves if you choose the number s at random from $0, \ldots, m$ but score s points only if $s \le r$. Then we have $a_m(n) = a(n)$, and

$$(m+1)a_r(n) = (m-r)a_r(n-1) + \sum_{k=0}^{r}(k + a_k(n-1)) =$$

$$(m-r+1)a_r(n-1) + \frac{r(r+1)}{2} + \sum_{k=0}^{r-1} a_k(n-1).$$

with $a_0(n) = 0$, $a_r(0) = 0$. So for $b_r(n) = r - a_r(n)$ we get the homogeneous equation

$$(m+1)b_r(n) = (m-r+1)b_r(n-1) + \sum_{k=0}^{r-1} b_k(n-1)$$

with initial condition $b_r(0) = r$. From this it is easy to deduce the following formula:

$$b_r(n) = \sum_{s=1}^{r} c_{rs} \left(\frac{m+1-s}{m+1}\right)^n$$

for some numbers c_{rs} determined recursively. For example, $b_1(n) = (\frac{m}{m+1})^n$, so $a_1(n) = 1 - (\frac{m}{m+1})^n$. In fact, it is easy to show by induction that $c_{rs} = 1$ for all r, s, so

$$b_r(n) = \sum_{s=1}^{r} \left(\frac{m+1-s}{m+1}\right)^n,$$

thus

$$a_r(n) = r - \sum_{s=1}^{r} \left(\frac{m+1-s}{m+1} \right)^n .$$

So the answer is

$$a(n,m) = m - \sum_{k=1}^{m} \left(\frac{k}{m+1} \right)^n$$

In particular, $a_m(\infty) = m$ (the average score if you play indefinitely).

G4. A street is lit by n street lights arranged in a row. If one of them burns out but its neighbors are still working[1], the Department of Public Works (DPW) does not do anything. However, once two consecutive lights are out of order, the DPW immediately replaces the light bulbs in all broken lights. For example:

$$\circ \quad \circ \quad \circ \quad \circ \quad \circ$$

$$\circ \quad \circ \quad \bullet \quad \circ \quad \circ$$

do nothing

$$\circ \quad \circ \quad \bullet \quad \circ \quad \bullet$$

still do nothing

$$\circ \quad \bullet \quad \bullet \quad \circ \quad \bullet$$

replace all

(i) What is the chance that the DPW will have to replace k lights, if lights break independently and with equal probability?

(ii) What is the average number of lights that they have to replace in each repair?

Compute the answers for $n = 9$ and $k = 4$ with two digits precision after the decimal point.

Solution. The chance that the arrangement does not require repair after m lights burn out is

$$r_m = \frac{\binom{n-m+1}{m}}{\binom{n}{m}}.$$

So the chance they have to replace k lights is

$$p_k = r_{k-1} - r_k.$$

The average number of lights they have to replace is

$$N = \sum k(r_{k-1} - r_k) = r_0 + r_1 + r_2 + \ldots$$

[1] Both neighbors, or only one if it is the first or the last light.

For $n = 9, k = 4$ we get

$$p_4 = \frac{25}{84} \approx 0.297619,\ N = \frac{93}{28} \approx 3.32.$$

G5. (i) Describe an algorithm to find the closed ball (disk) of smallest radius containing a given finite set of points (x_i, y_i), $i = 1, ..., n$, in $\mathbb{R}^2$.

(ii) Do the same for points (x_1, y_i, z_i), $i = 1, ..., n$, in $\mathbb{R}^3$.

(iii) Show that the ball in (i),(ii) is unique.

Solution. (iii) The ball is unique because the intersection of two distinct balls of the same radius is contained in a ball of smaller radius.

(i) Run through pairs of points (P, Q) and check if the circle with diameter PQ contains all other points inside. If so, we are done. Otherwise run through triples of points forming an acute-angled triangle and check if the circumscribed circle contains all other points. For one of them it must be so, and this is then the answer.

(ii) Same as (ii) but there is an extra step - need to run through quadruples of points whose convex hull contains the center of the circumscribed sphere.