Gonality Sequences of Multipartite Graphs

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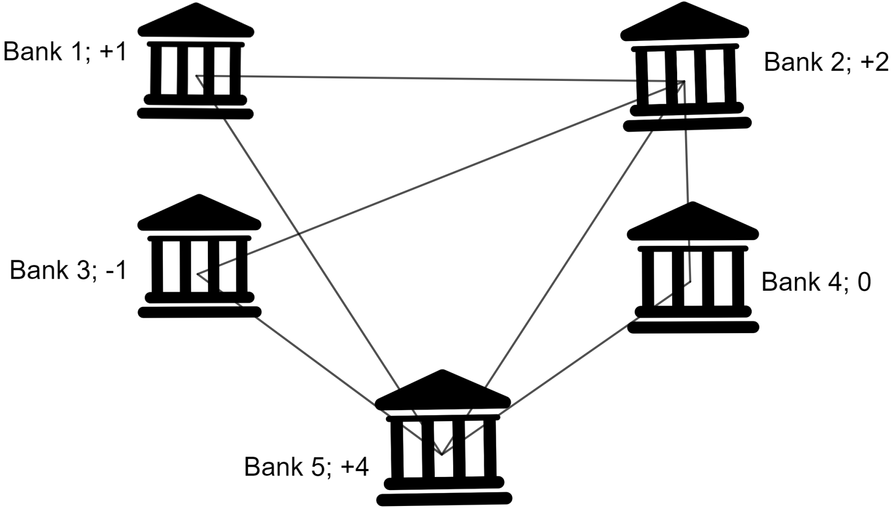
What is a Gonality Sequence?

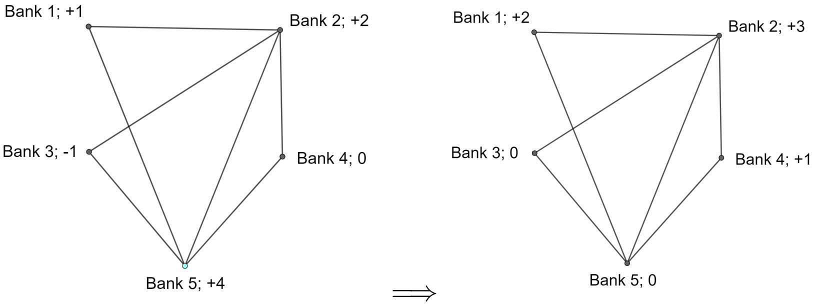
Let us begin with an analogy:

- Suppose we have a chain of banks
- Each of the banks has a current profit/loss value
- Some of these banks are connected to each other
- We can send money between the banks through these connections
- If a bank wants to send money, then it must send the same integer amount of money to all banks it is connected to
- A system of banks is called **effective** if it some banks can send some money so all the banks at least break even

An Illustration of Banks



An Illustration of Banks



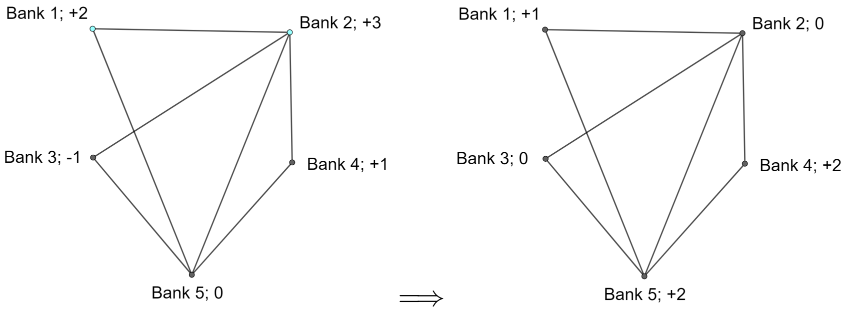
What is a Gonality Sequence?

Main Problem

Given that tomorrow, a total of k dollars in withdrawls will be made across all banks, what is the minimum total amount of money we must have total today so that they system of banks is effective for both today and tomorrow?

The amount of money needed for each integer $k \geq 0$ forms the gonality sequence for a given system of banks.

An Illustration of Gonality



Rephrasing in Terms of Chip-Firing

- We have a graph
- Each vertex has some integer number of chips on it
- We call this distribution of chips a **divisor**
- A firing move is a vertex giving a chip to each of its neighbors
- Divisors are equivalent to each other if reachable by firing moves
- A divisor is called **effective** if it is equivalent to a divisor with vertices having all nonnegative number of chips

Main Problem Rephrased

What is the minimum amount of chips an effective divisor can have given that we can subtract k chips off arbitrarily and the divisor will remain effective?

Computing Gonality Sequences

Question

How do we tell if a given divisor is effective or not?

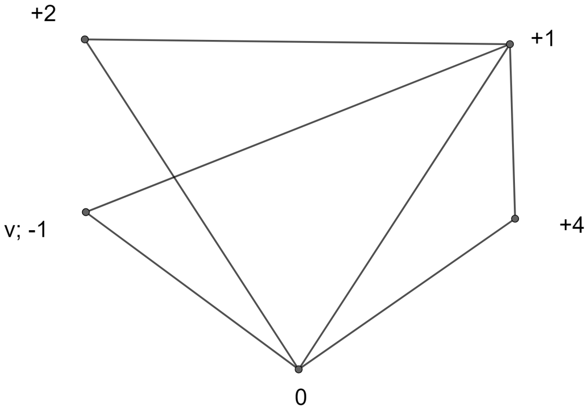
Answer

Dhar's Burning Algorithm!

Dhar's Burning Algorithm

Step 1

Fix a vertex v of the graph, and find an equivalent divisor that is effective away from v, or nonnegative at every vertex other than v.



Dhar's Burning Algorithm

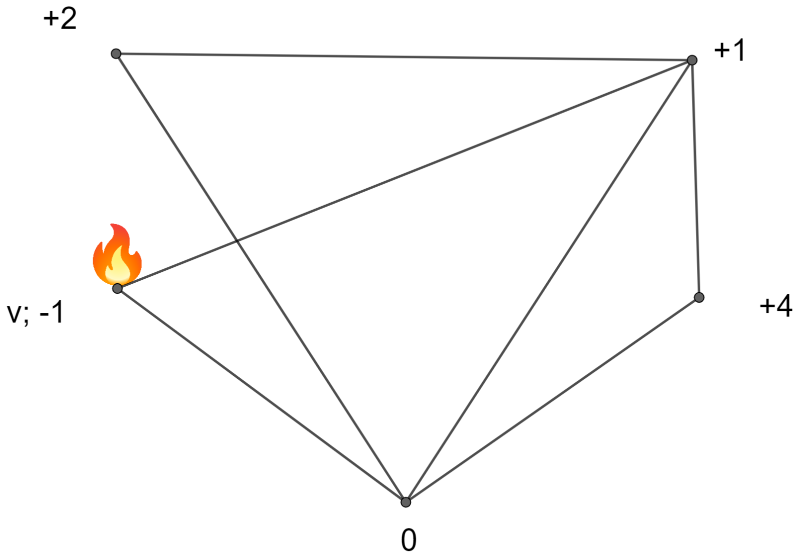
Step 2

Burn the vertex v. Burning means marking the vertex.

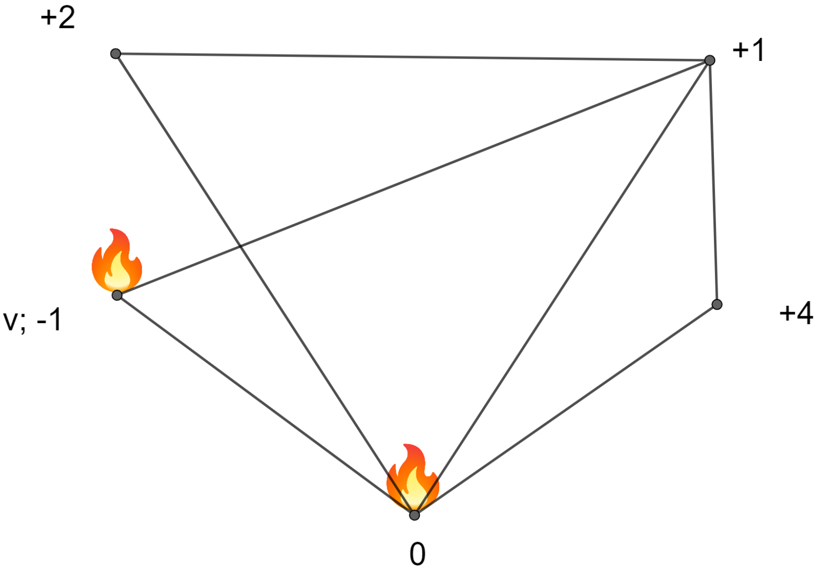
Step 3

Go through the remaining vertices, and burn those that have more burnt neighbors than chips. Repeat this step until no more vertices burn.

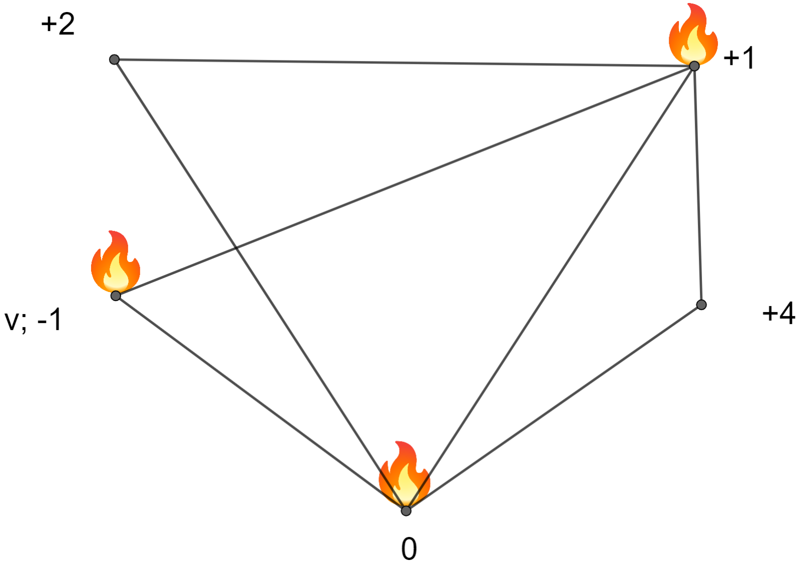
Steps 2 and 3 Illustration



Steps 2 and 3 Illustration



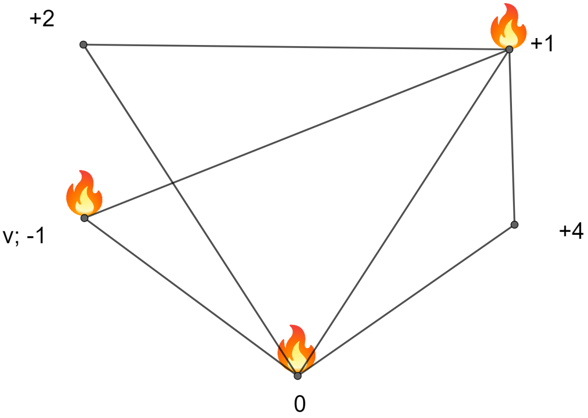
Steps 2 and 3 Illustration

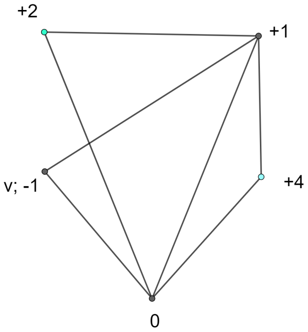


Dhar's Burning Algorithm

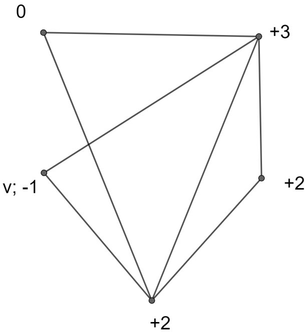
Step 4

If there are no unburnt vertices, move on to step 5. Otherwise, fire all the unburnt vertices, and go back to step 2.





$\Longrightarrow$



Dhar's Burning Algorithm

Step 5

If v has a nonnegative number of chips, the original divisor was effective. Otherwise, it is not.

Computing Gonality Sequences through Computers

Question

How do we compute gonality sequences for certain graphs?

- Given a graph, compute the genus $g = E - V + 1$
- The end of the gonality sequence ($k \geq g$) is known.
- Now that we know when a divisor is effective, we can brute-force compute the gonality sequence for graphs of small genus
- This allows us to try and form conjectures, which we can then try and prove combinatorially

Verifying Conjectures for Gonality Sequences

Question

How do we verify conjectures generated by the computer?

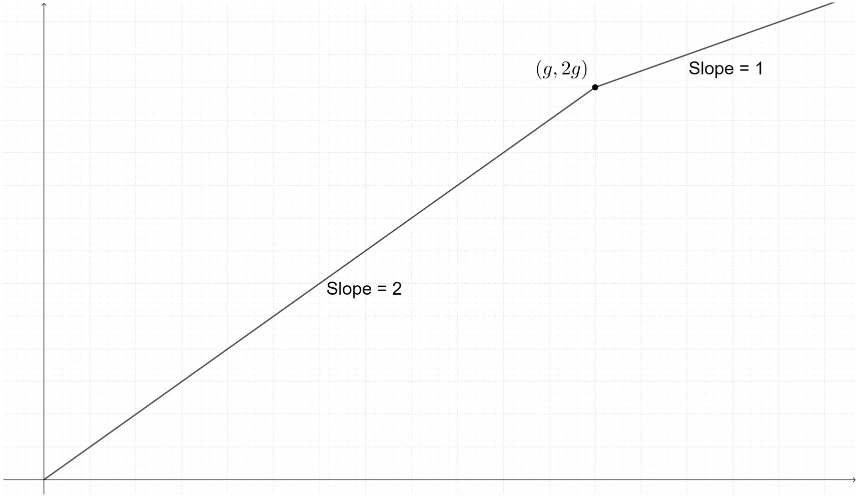
Answer

Bounding! Lots of bounding!

- Riemann-Roch for Graphs
- Clifford's Theorem
- Finding specific divisor examples

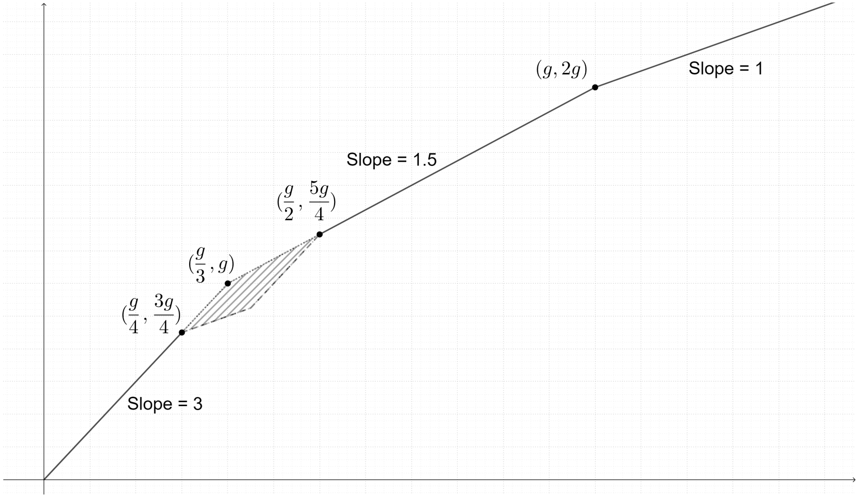
Results

For the tripartite graphs $K_{n,1,1}$:



Results

For the 4-partite graphs $K_{n,1,1,1}$:



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