Arrangements of Simplices in Fine Mixed Subdivisions

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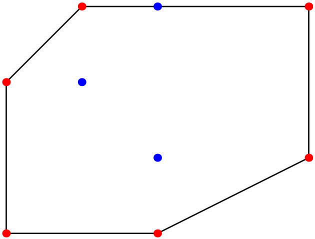
10/15/22

Polytopes

Definition

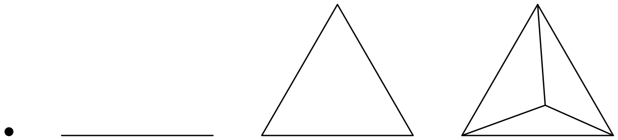
A *polytope* is the convex hull of a set of points $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. A point $\mathbf{v}_i$ is a *vertex* of the polytope if removing it from the set of points changes the polytope.



Simplices

Definition

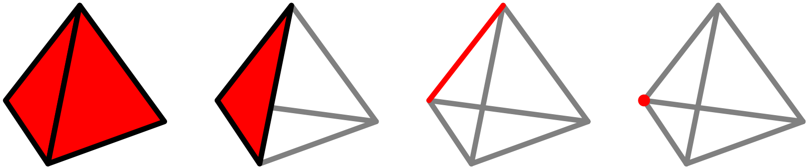
A d-dimensional *simplex* is a polytope in d dimensions with exactly $d + 1$ vertices which don't all lie on the same hyperplane.



Faces

Definition

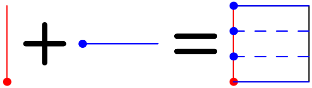
The *faces* of a polytope are the intersections of the polytope with any hyperplane that doesn't cut through it. (These include vertices, edges, the empty set, and the whole polytope itself.)

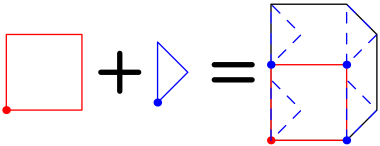


Minkowski Sum

Given two sets of vectors P and Q, their Minkowski sum is defined to be

$$P + Q := \{\mathbf{p} + \mathbf{q} : \mathbf{p} \in P, \mathbf{q} \in Q\}.$$





Fine Mixed Cells

Consider the regular simplex $\triangle_{d-1}$ embedded in d dimensions with vertices $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d$. (Note that this simplex is $(d-1)$-dimensional.)

Definition

A *fine mixed cell* is a $(d-1)$-dimensional polytope of the form

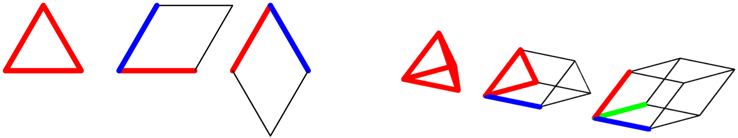
$$B_1 + B_2 + \cdots + B_n,$$

where the B_i are faces of $\triangle_{d-1}$ such that

$$\sum_{i=1}^{n} \dim(B_i) = d - 1.$$

Examples of Fine Mixed Cells

Below are some examples of fine mixed cells in 2 and 3 dimensions. The faces which are being summed are outlined in different colors.

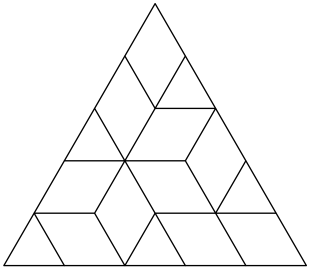


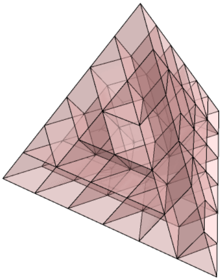
Note that fine mixed cells cannot be rotated or translated arbitrarily. We translate them by making some of their components vertices.

Fine Mixed Subdivision

Definition

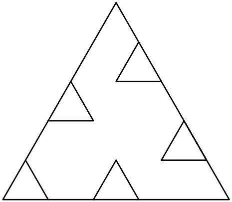
A *fine mixed subdivison* of $n\triangle_{d-1}$ is a partition of $n\triangle_{d-1}$ into fine mixed cells such that any two cells intersect at a face of both (possibly empty).

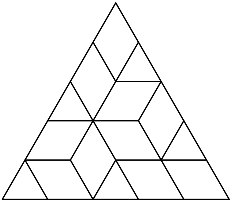


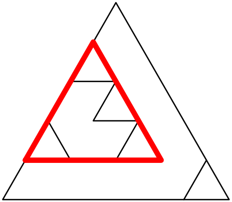


Spread-Out Condition

It's known that a fine mixed subdivision must have exactly n cells that are simplices. Furthermore, any smaller simplex of size k contains at most k cells which are simplices (Ardila–Billey, 2007). Call an arrangement of n simplices that satisfies these conditions *spread-out.*







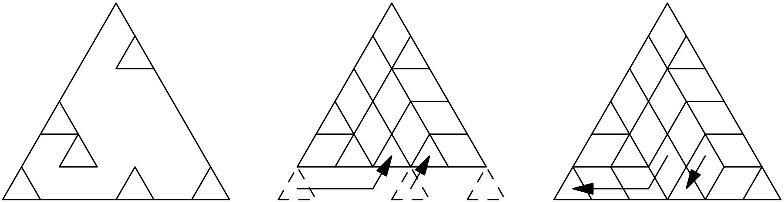
Main Question

Spread-Out Simplices Conjecture (Ardila–Billey, 2007)

Given a spread-out arrangement of n simplices in $n\triangle_{d-1}$, does there always exist a fine mixed subdivison with simplices only at these positions?

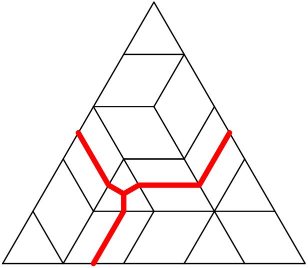
2 Dimensions

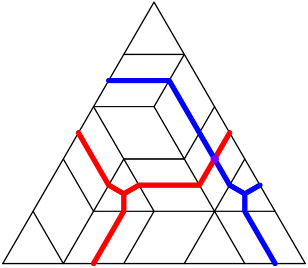
This problem has already been solved in the $d-1=2$ case. Below we outline an inductive proof which involves "sliding" every triangle in the bottom row except one into the next row (Ardila–Billey, 2007).



Tropical Pseudolines

In 2 dimensions, we can define a *tropical pseudoline* from each simplex, as depicted on the left. (We will refer to these as just pseudolines.)

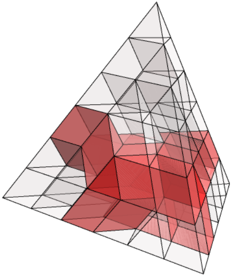


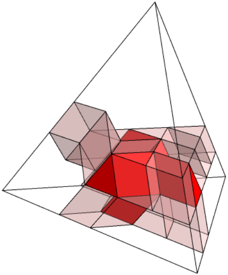


- Any two pseudolines intersect at exactly one point.
- Pseudolines behave like actual lines.
- Pseudolines show the combinatorial structure of a subdivision.

Tropical Pseudoplanes

We can define *tropical pseudoplanes* similarly in 3 dimensions, though they are much more complex.

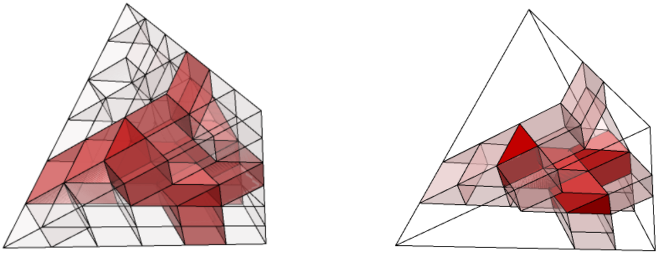




- Any three pseudoplanes intersect at exactly one point.
- Pseudoplanes behave like actual planes.
- Pseudoplanes show the combinatorial structure of a subdivision.

Tropical Pseudoplanes, continued

Here's the same tiling and pseudoplane, from a different perspective.



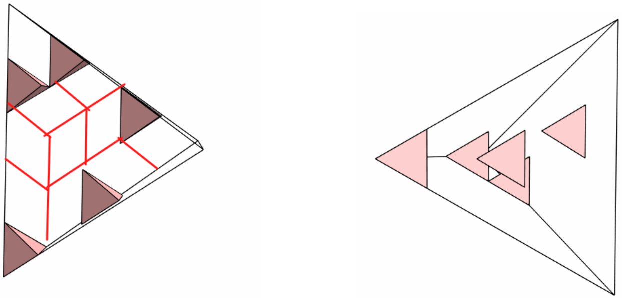
Main Result

Theorem (L.–Yao, 2022)

Given an arrangement of n tetrahedrons in $n\triangle_3$, consider the triangle formed by an edge of the $n\triangle_3$ and the midpoint of the opposite edge. If the projection of the tetrahedra onto this triangle creates a spread-out configuration of triangles, so that it can be tiled by a 2-dimensional tiling T. Then, there exists a fine mixed subdivision with those tetrahedra where each cell in the tiling projects to a cell of T.

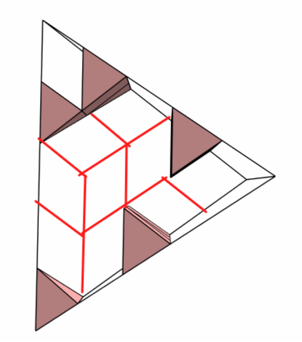
Example

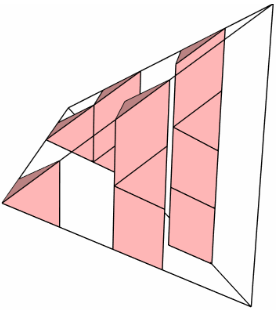
Here's an example of such an arrangement, with the 2D tiling drawn in.



Building Columns

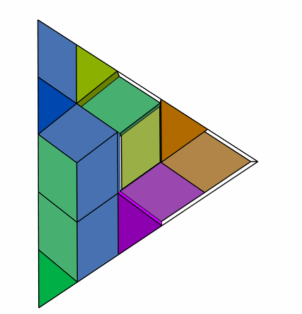
First, we build "columns" on each tetrahedron.

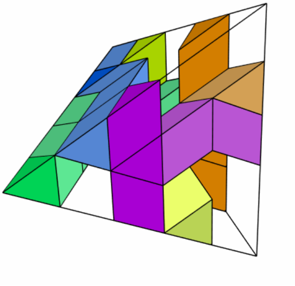




Extending Pseudolines

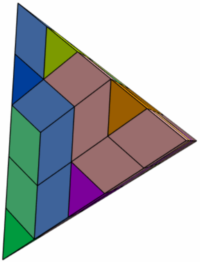
Next, we extend pseudolines rightwards from each tetrahedron.

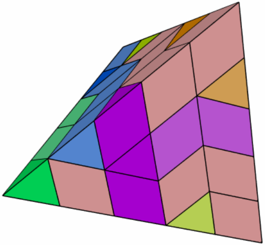




Completing the Subdivision

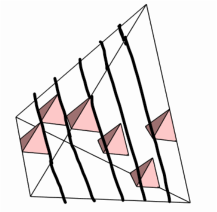
Finally, we fill the remaining "columns" with parallelpipeds.

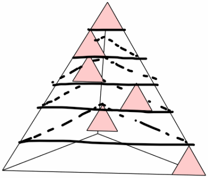


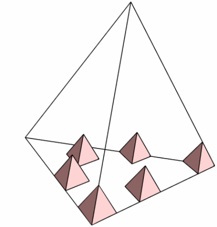


Corollaries

This theorem covers the three special cases of one tetrahedron in each edge layer, one in each face layer, and all on one face, as shown below.







Future Directions

In the future, we hope to resolve the $d-1=3$ case fully, then work on the conjecture in for dimensions $d-1 \geq 4$.

We have a few different directions to explore. For example, we can attempt an inductive argument, like in 2 dimensions, where we slide tetrahedra to more favorable placements.

As an alternative approach, we can try to cut the tetrahedron into pieces, each of which can be tiled by the methods we mentioned before.

Acknowledgements

I would like to thank my mentor, Yannick Yao, for introducing me to this question and helping me make progress on it.

I would also like to thank Dr. Etingof, Dr. Gerovitch, Dr. Khovanova, and the PRIMES-USA Program for providing me this valuable opportunity to conduct mathematical research.

References

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[2] Frederico Ardila and Mike Develin, Tropical hyperplane arrangements and oriented matroids, *Mathematische Zeitschrift*. **262** (2008), no. 4, 795–816

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[4] Günter M. Ziegler, *Lectures on Polytopes*, Springer Graduate Texts in Mathematics (1995).