# New Properties of the Intrinsic Information and Their Relation to Bound Secrecy

Andrew Tung, Karthik Vedula

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# Overview

1. Entropy

2. Secret-key rate and bound secrecy

3. Our result

# Informal definition

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- Encoding a variable using bits = replacing outputs with binary strings (used to compress data)

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Ex: Encoding from $a \to 0$, $b \to 01$. Bad because if first digit $= 0$, then we're confused

Goal: Try to use as few bits (on average) as possible to encode without confusion

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Goal: try to find best prefix code

**Key Point**

Entropy is minimum number of bits (on average) needed to prefix encode a variable

# Motivating example

Consider a random variable defined as[1]

$$X = \begin{cases} a & \text{probability } \frac{1}{2} \\ b & \text{probability } \frac{1}{4} \\ c & \text{probability } \frac{1}{8} \\ d & \text{probability } \frac{1}{8} \end{cases}$$

How many bits do you need to encode this information?

[1] Example from Nielsen and Chuang, "Quantum Computation and Quantum information."

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Encode $a \to 00$, $b \to 01$, $c \to 10$, $d \to 11$. Prefix code because all code words are distinct, fixed length

**Key Point**

Entropy of variable with $n$ outputs $\leq \log_2(n)$.

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Average number of bits required:

$$\frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 3 + \frac{1}{8} \cdot 3 = \frac{7}{4} < 2$$

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Check: $H(X) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{4}\log\frac{1}{4} - \frac{1}{8}\log\frac{1}{8} - \frac{1}{8}\log\frac{1}{8} = \frac{7}{4}$.

# Operational motivation

**Theorem (Shannon's noiseless coding theorem)**

Given a random variable $X$, any encoding using less than $H(X)$ bits on average is not reliable, while there is always an reliable encoding using $H(X) + \epsilon$ bits on average for all $\epsilon > 0$.

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**Key Point**

Shannon entropy = our notion of entropy

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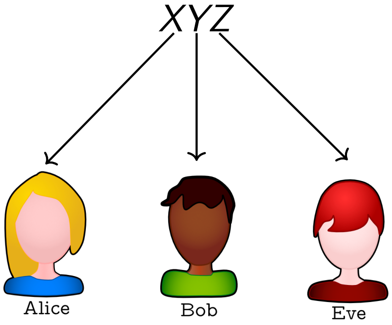

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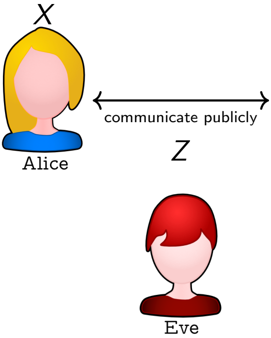


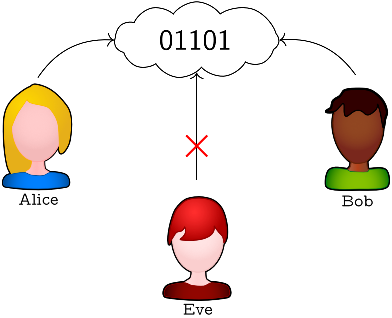

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**Theorem (Maurer & Wolf, 1999)**

If Alice and Bob do not share secrecy, they cannot distill a secret key.

# Examples

| $X$ \ $Y$ | 0 | 1 |
|---|---|---|
| 0 | 1/4 | 1/4 |
| 1 | 1/4 | 1/4 |

| $Z$ | prob. |
|---|---|
| 0 | 1/2 |
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| Share secrecy | Can gen. key |
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| ✗ | ✗ |

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Share secrecy: ✗ Can gen. key: ✗

**Example 2**

| $X$ \ $Y$ | 0 | 1 |
|---|---|---|
| 0 | 1/2 | 0 |
| 1 | 0 | 1/2 |

| $Z$ | prob. |
|---|---|
| 0 | 1/2 |
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Share secrecy: ✓ Can gen. key: ✓

**Example 3**

| $X$ \ $Y$ | 0 | 1 |
|---|---|---|
| 0 | 1/2 | 0 |
| 1 | 0 | 1/2 |

Eve receives what Alice gets.

Share secrecy: ✗ Can gen. key: ✗

# Bound secrecy

The conjecture of bound secrecy states that there are distributions $XYZ$ such that Alice and Bob share secrecy but they cannot agree on a secret key.

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This seems impossible!

# Another non-example

| $X$ / $Y$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | $1/8$ | $1/8$ | 0 | 0 |
| 1 | $1/8$ | $1/8$ | 0 | 0 |
| 2 | 0 | 0 | $1/4$ | 0 |
| 3 | 0 | 0 | 0 | $1/4$ |

$$Z \equiv X + Y \text{ mod } 2 \text{ if } X, Y \in \{0, 1\},$$
$$Z \equiv X \text{ mod } 2 \text{ if } X, Y \in \{2, 3\}$$

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$$Z \equiv X + Y \bmod 2 \text{ if } X, Y \in \{0, 1\},$$
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Share secrecy ✓

Can generate a secret key ✓

How do Alice and Bob extract the secret key?

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Let $U = \lfloor X/2 \rfloor$. This is a secret bit shared between Alice and Bob.

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Let $U = \lfloor X/2 \rfloor$. This is a secret bit shared between Alice and Bob.

If Eve knew $U$, Alice and Bob would have no secrecy.

# Our results

Formalizing the previous example:

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The *reduced intrinsic information* is informally the smallest amount of information we need to tell Eve in order for Alice and Bob to share no secrecy.

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The *reduced intrinsic information* is informally the smallest amount of information we need to tell Eve in order for Alice and Bob to share no secrecy.

**Our results**

Assuming the conjecture of bound secrecy, we have shown that the reduced intrinsic information does NOT measure whether Alice and Bob can agree on a secret key.

# Acknowledgements


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*Thanks for listening!*