Algorithm Analysis

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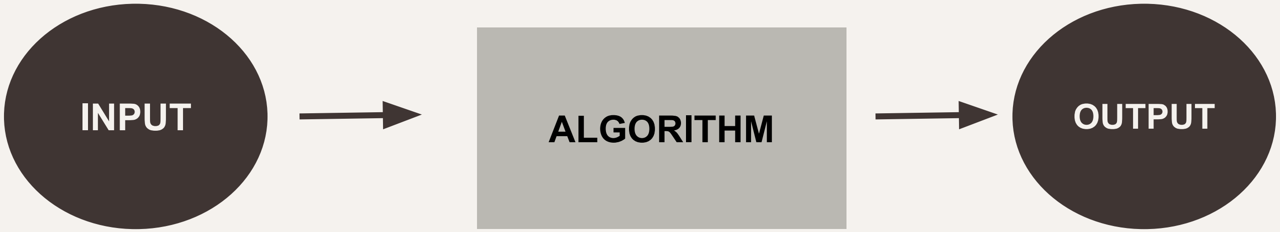
Intro to Algorithm Analysis

Algorithm: A set of instructions that a computer follows and applies on input

- **Input:** data entered into an algorithm
- **Output:** results produced

Algorithm Analysis: How long it takes for the computer to follow these instructions

- Best measured in terms of large input sizes



Measuring Efficiency

- Can't measure time it takes to run ~ machine dependent

Need:

- Machine Independence
- How algorithm behaves as input size increase

Run Time:
# of steps or operations executed
Depends on input size (#elements)

Input Size:
#elements inserted in algorithm

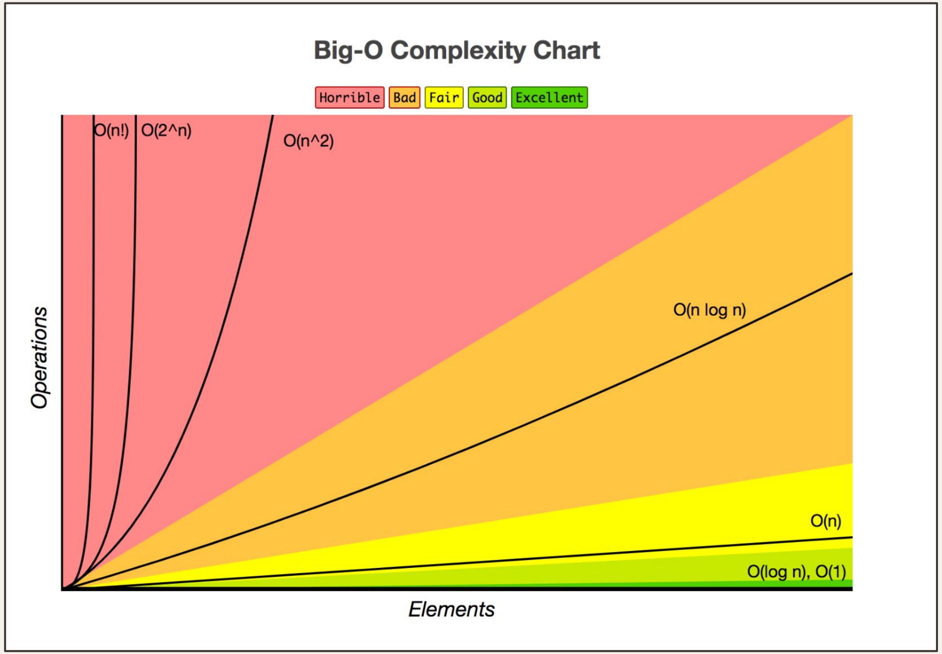
Represented by *n*

Cases to consider:

- Worst-case
- Best-case
- average case

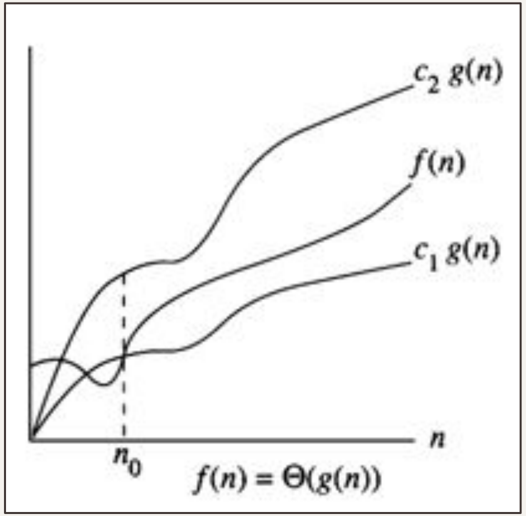
Big O notation *O(n)*

- describe the upper-bound
- worst-case scenario



Big-Omega Ω

- lower bound
- best-case scenario

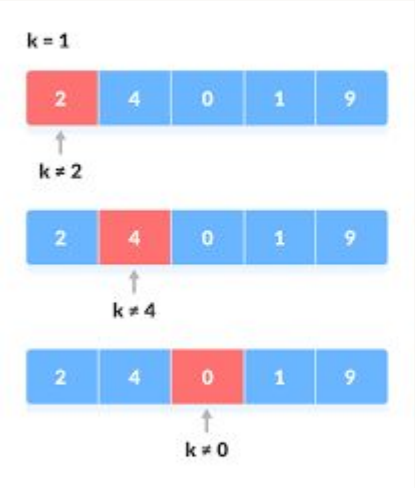


Theta Θ

- describes best and worst case scenario.
- gives the exact bound.

Search Algorithms

Linear Search



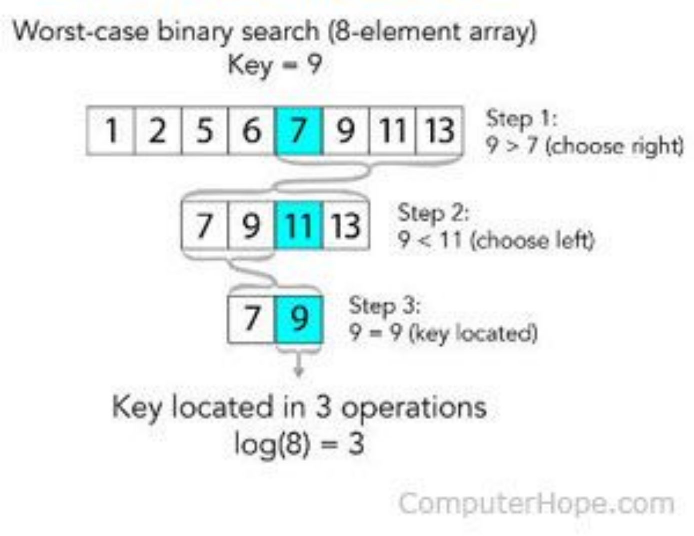
O(n)

Linear Search: Sort through until desired element is found

Binary Search:

- Divides data set in half
- Compares target value with middle term
- Eliminates half set that does not contain T
- Repeats until T found

Binary Search



$O(\log_2 n)$

Recursive Algorithms

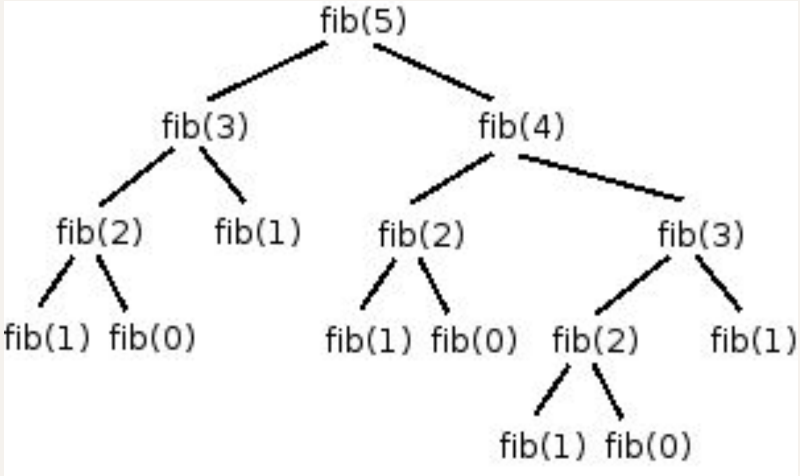
- Divide the larger problem into subproblems by calling itself
- Divides until the base case is reached and performs the algorithm’s objective on easily solvable inputs

Fibonacci Sequence

$$f(n)=\begin{cases} 0 & if \quad n=0 \\ 1 & if \quad n=1 \\ F(n-1)+F(n-2) & if \quad n>1 \end{cases}$$

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ...

Each number is the sum of the previous two numbers.



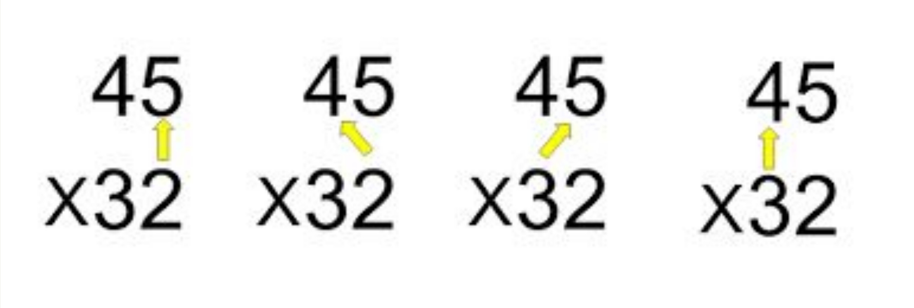
Karatsuba Algorithm

- Fast Multiplication Algorithm

- Reduces time it takes to multiple 2 even *n*-digit numbers
    - If odd, add zeros

Naive Method of Multiplying

- Each digit in *x* multiplied to each digit in *y*
- Total: 4 single-digit computations to find x *y
- **n^2 operations** (*n* → # digits in each number)



n^2 operations → Time Complexity $O(n^2)$

Deriving Karatsuba Algorithm

1. Split number in half

x= **45** = 40 + 5

y = **32** = 30 + 2

High bit: 4 = *a*
Low bit: 5 = *b*

High bit: 3 = *c*
Low bit: 2 = *d*

2. Another way to express multiplication

xy = (40 +5) x (30+2)

3. Distributive property

xy **= (40*30) + (40*2)+ (5*30) + (5*2)**

Even this way, still 4 computations (n^2)

n → #digits in each number

Karatsuba Algorithm

x= **45** = 40 + 5

y = **32** = 30 + 2

High bit: 4 = *a*
Low bit: 5 = *b*

High bit: 3 = *c*
Low bit: 2 = *d*

***xy* = (40*30) + (40*2)+ (5*30) + (5*2)**

(40*30)	+	(40*2 + 5*30)	+	(5*2)
High bits		***Middle*** bits		***Low*** bits
ac	+	***(ad + bc)***	+	***bd***

KA divides problem into 3 sub-problems (instead of 4)

Karatsuba Algorithm Cont.

x= **45** = 40 + 5

High bit: 4 = *a*
Low bit: 5 = *b*

y = **32** = 30 + 2

High bit: 3 = *c*
Low bit: 2 = *d*

***xy* = (40*30) + (40*2)+ (5*30) + (5*2)**

(40*30) + (40*2 + 5*30) + (5*2)

High bits + ***Middle*** bits + ***Low*** bits

ac + ***(ad + bc)*** + ***bd***

Middle Term

***(ad + bc)* =** *(a + b)(c + d) − ac − bd*

Proof → Gauss' Trick:

(~~a~~c + ad + bc + ~~b~~d) - ~~a~~c - ~~b~~d

Subtract high & low bits from total to find middle

NOTE*: Bases of ten (zeros) can be ignored for now and added on at the end*

Generalized Karatsuba Algorithm

$$x.y = (10^n ac + 10^{n/2}(ad + bc) + bd$$

H **M** **L**

1. Recursively compute ac ***High bits***
2. Recursively compute bd ***Low bits***
3. Recursively compute (a+b)(c+d) = ac+bd+ad+bc ***Middle bits***

Gauss' Trick : (3) – (1) – (2) = ad + bc

NOTE*: Bases of ten (zeros) can be ignored for now and added on at the end*

Karatsuba Run Time

(4*3) + (4+2) * (5+3) + (5*2)

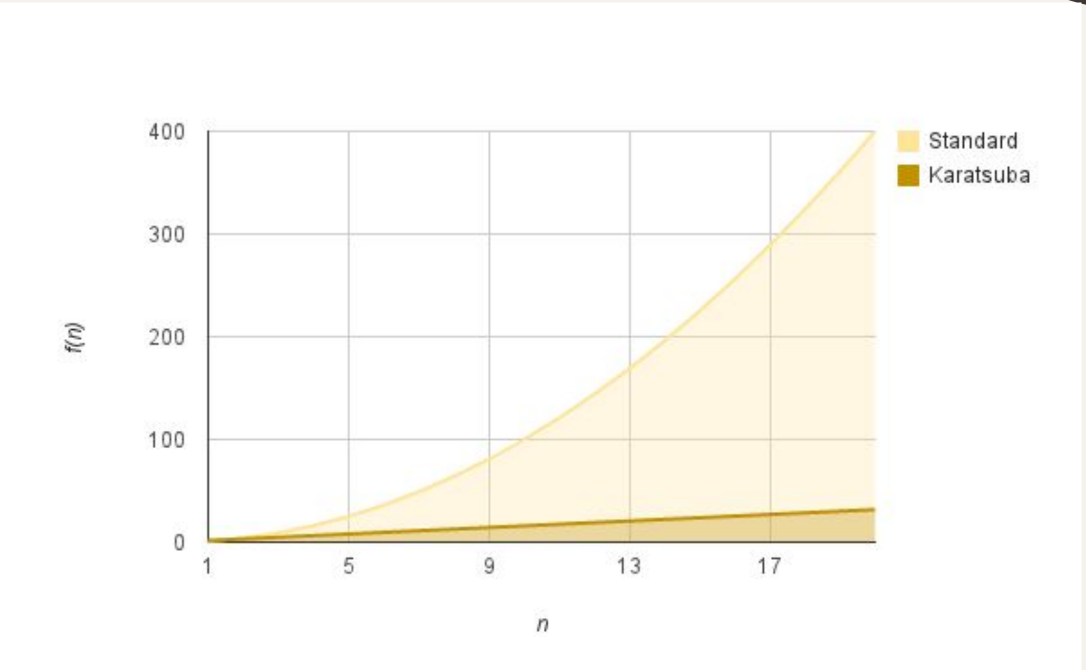
High bits	Middle bits	Low bits
ac	***(a + b)(c + d) − ac − bd***	***bd***
n/2 * n/2	**(n/2) * (n/2)**	**n/2 * n/2**
1 single-digit computation	1 single-digit computation	1 single-digit computation

Total: 3 computations instead of 4

Time Complexity

$$T(n) = 3T\left(\frac{n}{2}\right) + O(n).$$

$$\Theta\left(n^{\log_2 3}\right) \approx \Theta\left(n^{1.585}\right).$$



T → *run time for multiplication*
O(n) → *standard time for arithmetic*

Akra - Bazzi

Akra Bazzi Method

Recurrence relation: expression of a term as a function of the terms before it.

$$T(x) = g(x) + \sum_{i=1}^{k} a_i T(b_i x + h_i(x))$$

Takes recurrence relation as input: outputs asymptotic time complexity.

$$\sum_{i=1}^{k} a_i b_i^p = 1$$

$$T(x) = \Theta\left(x^p \left(1 + \int_1^x \frac{g(u)}{u^{(p+1)}} du\right)\right)$$

Strassen Algorithm

Strassen Algorithm

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \times \underbrace{\begin{bmatrix} e & f \\ g & h \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}}_{C}$$

$$M_1 = (A_{11} + A_{22})(B_{11} + B_{22});$$
$$M_2 = (A_{21} + A_{22})B_{11};$$
$$M_3 = A_{11}(B_{12} - B_{22});$$
$$M_4 = A_{22}(B_{21} - B_{11});$$
$$M_5 = (A_{11} + A_{12})B_{22};$$
$$M_6 = (A_{21} - A_{11})(B_{11} + B_{12});$$
$$M_7 = (A_{12} - A_{22})(B_{21} + B_{22}),$$

$$= \begin{pmatrix} m_1 + m_4 - m_5 + m_7 & m_3 + m_5 \\ m_2 + m_4 & m_1 + m_3 - m_2 + m_6 \end{pmatrix}$$

T(x) = 7T(x/2) + Θ(n³)

Proof: Akra Bazzi

$$T(x) = 7T(x/2) + \Theta(n^3)$$

$a = 7$
$b = ½$
$p = \log 7$

Works Cited

Cormen, Thomas H., et al. Introduction to Algorithms. MIT Press; McGraw-Hill, 1990.

Bender, Edward A., and Williamson, Stanley G. Mathematics for Algorithm and Systems Analysis. Courier Corporation, 2005.

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