An Introduction to Matrix Transformation Groups

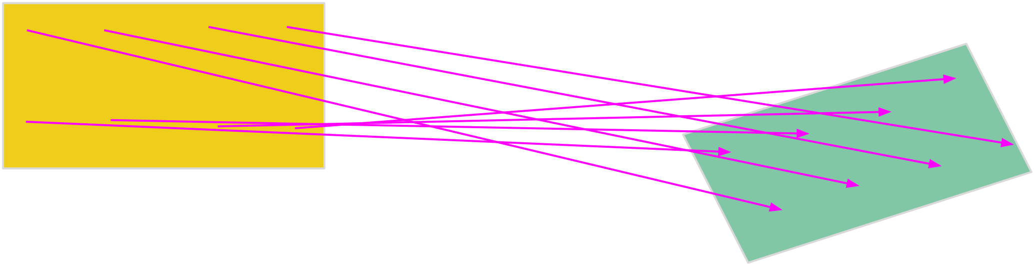
By Kevin Zhao (Mentor: Sanjay Raman)

Matrix Basics

Assume all matrices are square.

- Determinant of n x n matrix = n-dimensional volume of the parallelepiped spanned by column vectors
- A square matrix has nonzero determinant if and only if it is invertible.
- Invertible matrices consist of **Linearly Independent** columns
  - All points in the range of the matrix transformation are a result of a unique point in the domain

An example is shown below of a possible map.



Matrix Transformations

- We can specify linear transformations using matrices
- Three basic types of linear transformation: Rotation, Reflection, and Dilation.

Definition 4.1. A rotation matrix M describes a rotation about an axis through an angle θ. If $\theta = \frac{2\pi}{n}$, then $M^n = I$.

Definition 4.2. For a reflection, there is a basis in which the reflection matrix has all non-diagonal terms as zero, and diagonal terms as ± 1 depending on the plane the reflection is based off of.

Definition 4.3. A dilation is given by a matrix λI, where λ is the scale factor.

Note that if M is a rotation of a angle $\theta\pi$ such that θ is rational, there always exists an integer n such that $M^n = I$.

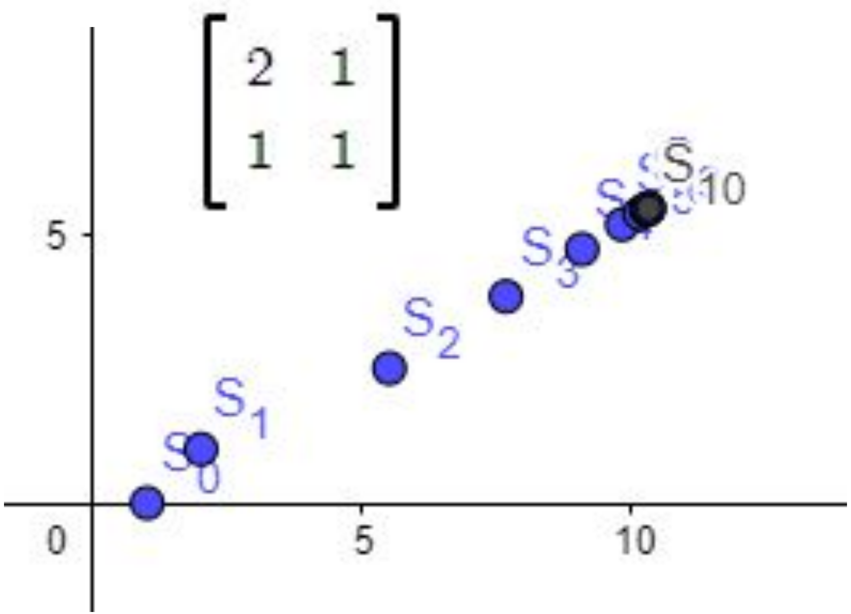
The Exponential and Logarithmic Functions for Matrices

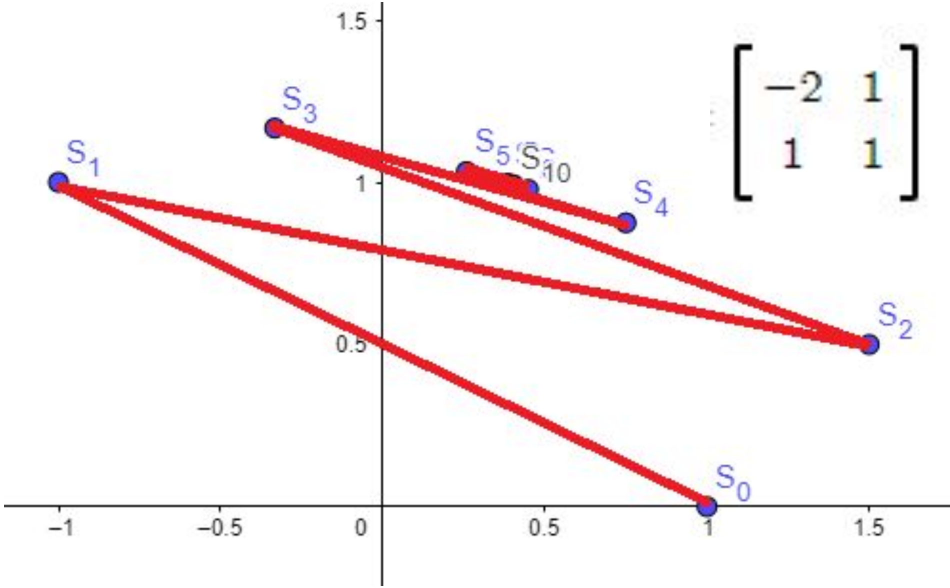
- To describe rotations, we use the *Matrix Exponential*
- The key definition is the following: **The exponential function, Exp(X), and the logarithm function, Log(X), are defined by power series.**
- The definition of the Exp and Log functions, in mathematical terms, are:

$$Exp(X) = \sum_{n \geq 0} \frac{1}{n!} X^n \text{ and } Log(I + X) = \sum_{n \geq 0} \frac{(-1)^{n-1}}{n!} X^n.$$

The Exponential and Logarithmic Functions for Matrices (cont.)

These are plots of what Exp(X) might look like for a two-dimensional matrix acting on the vector <1, 0>, with the S values corresponding to the number of terms in the summation used. Note the convergence of Exp.

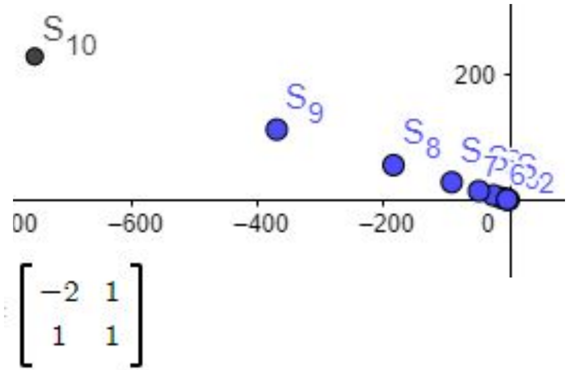


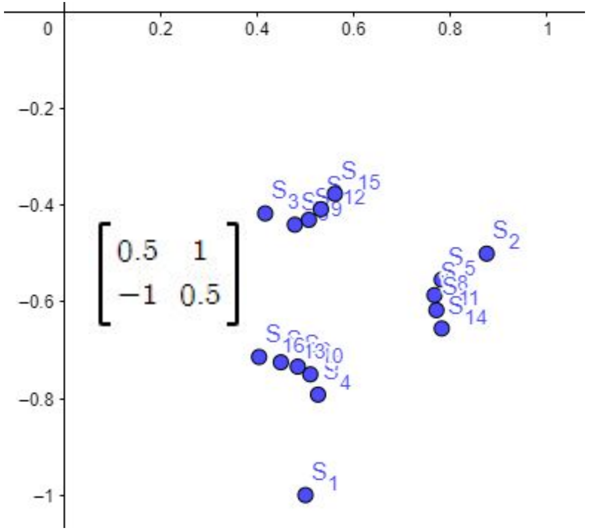


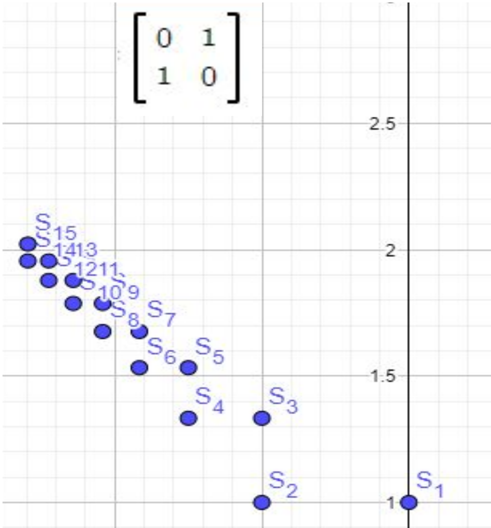
Logarithmic Summation Plots of Matrices

For logarithmic plots, they are much different. Some sequences converge while others diverge.

These are plots of what Log(1+X) might look like for a two-dimensional matrix acting on a unit vector, with the S values corresponding to the number of terms in the summation used.







Exponential Function by Addition and Power Series

To check the theory behind these definitions, let's first take Exp(X). We note that Exp is obviously closed under addition of multiples of X, and multiplying gets

$$\mathrm{Exp}(aX)\mathrm{Exp}(bX) = \left(\sum_{k=0}^{\infty} \frac{1}{k!} \cdot a^k X^k\right)\left(\sum_{k=0}^{\infty} \frac{1}{k!} \cdot b^k X^k\right).$$

So, for all X^n terms, the coefficient we have is going to be $\sum_{k=0}^{n} \frac{a^k b^{n-k}}{k!(n-k)!}$. Each term is the $a^k b^{n-k}$ term in $(a+b)^n$ for all k, so we have that

$$\sum_{k=0}^{n} \frac{n! a^k b^{n-k}}{k!(n-k)!} = (a+b)^n \to \sum_{k=0}^{n} \frac{a^k b^{n-k}}{k!(n-k)!} = \frac{(a+b)^n}{n!}$$

meaning that

$$\mathrm{Exp}(aX)\mathrm{Exp}(bX) = \sum_{k=0}^{\infty} \frac{(a+b)^k}{k!} \cdot X^k = \mathrm{Exp}((a+b)X).$$

An intuitive reason on why this works is because of Maclaurin series upon e^x - because we know that $e^{a+b} = e^a e^b$ and Exp is just powers of e with respect to matrices.

Diagonalizing a matrix: Exp and Log

As shown below, if a matrix $M = P^{-1}DP$ for some invertible matrix P, we have an interesting equation for Exp(M^k).

We notice that

$$Exp(M^k) = I + M^k + \frac{M^{2k}}{2!} + \frac{M^{3k}}{3!} + \ldots = I + \frac{P^{-1}D^kP}{1!} + \frac{P^{-1}D^{2k}P}{2!} + \ldots$$

Now, we know from the previous diagonal matrix identity, that because matrices are distributive, we can break them up:

$$Exp(M^k) = \left(P^{-1} + P^{-1}D^k + \frac{P^{-1}D^{2k}}{2!} + \frac{P^{-1}D^{3k}}{3!} + \ldots\right) P$$

$$= P^{-1}\left(I + D^k + \frac{D^{2k}}{2!} + \frac{D^{3k}}{3!} + \ldots\right) P = P^{-1}Exp(D^k)P.$$

This lets us arrive at an interesting equation: Since $M^k = P^{-1}D^kP$, then $Exp(P^{-1}D^kP) = P^{-1}Exp(D^k)P$.

Representing Rotation Matrices using the Matrix Exponential

- Note that Exp(Log(R))^k = Exp(kLog(R)) where R is a rotation matrix and k is a nonnegative integer.
- Only the magnitude of the argument of Exp in the direction of Log (R) varies
- Hence, for a single input k, we can determine a rotation matrix R. k uniquely defines the rotation angle.
- We can uniquely compute this angle (done in the next slide)
- This rotation group is SO(2), the two-dimensional rotation group. This is a simple example of a **Lie Group**.

Representing Rotation Matrices using the Matrix Exponential (cont.)

Here, we are uniquely expressing the Log of a rotation matrix as a scalar, given by the angle of rotation, times a constant matrix.

We first write the 90 degree counterclockwise rotation matrix R, and then proceed to find its Log with our definitions.

With motivation from 4-cycles of the derivatives of sin and cos since if the fourth derivatives of sin and cos are themselves, we first notice that if we let the rotation matrix R represent

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

then we note that from above,

$$Exp(R) = \sum_{i=0}^{3} \sum_{j=0}^{\infty} \frac{R^{i \pmod 4}}{(4j+i)!}$$

which, since

$$R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, R^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \text{ and } R^4 = I,$$

we see that we can rewrite each element in the matrix as (since the Maclaurin series revolves around the number 1 here) the following matrix, with a rotation angle:

$$Exp(R) = \begin{bmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{bmatrix} \to R = Log\left(\begin{bmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{bmatrix}\right)$$

Representing Rotation Matrices using the Matrix Exponential (cont.)

A more general expression is shown below:

$$\mathrm{Exp}\left(\theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\mathrm{Log}\left(\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\right) = \theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

As we noted earlier, the series for Log does not always converge, so the second formula holds only when the angle is sufficiently small.

Lie Groups: A Generalization of Continuous Matrix Groups

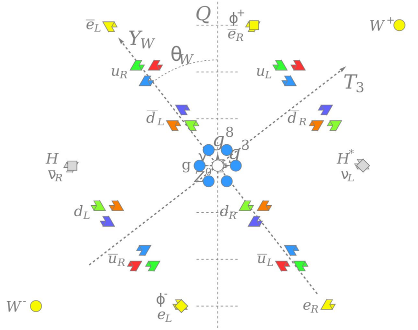


Diagram for interaction of particles
[Source: Wikipedia]

- A generalization of the study of matrix groups
- Relation to multiple other subjects
  - Geometry - transformations are a big part of Lie Theory
    - Projective Linear Groups & Cross Ratio
  - Physics - shows physical system especially particle physics
  - Linear Algebra
- **Our previous slides about the matrix exponential and logarithmic functions have discussed a specific realization of the exponential map in Lie Group Theory.**

Final Remarks

- Previously, we identified matrix transformations individually with respect to the rotation and Exp and Log functions.
- With Lie groups, we understand that our previous matrix groups have been subgroups of the **General Linear (GL)** group over some vector space
- More generally, Lie groups do not always have such a structure!
- The exponential map produces a homomorphism from the additive group real numbers to a “rotation group”, which is a known as a **one-parameter subgroup** of a more general Lie group. This, in Lie Theory, can generalize the map of rotations.

Acknowledgements

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Thanks for listening to my presentation!

References

- “Introduction to Lie Groups and Lie Algebras” by Alexander Kirillov, Jr.
- “An introduction to matrix groups and their applications” by Andrew Baker