Group Theory

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MIT PRIMES Circle

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Introduction

Jaeyi

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Hello, my name is Jaeyi Song and I am a freshman.

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Sophia

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Hi, my name is Sophia Hou and I'm a sophomore.

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One fun fact is that I have a younger sister.

Groups

Introduction to Groups

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Definition

A **group** $(G, +)$ is a set G with a binary operation $*$ that has three requirements satisfied:

1. Associativity: $a * (b * c) = (a * b) * c$ for all elements $a, b, c \in G$.
2. Identity: there is an element $e \in G$ in which $a * e = e * a = a$ for all elements of G. The identity for groups under multiplication is 1, under addition it is 0.
3. Inverse: For an element $a \in G$, there is the inverse of a (let's say b) that satisfies $a * b = b * a = e$.

Example 1

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This is an example of a cyclic group, which is a special type of group in which every element can be written as iterated copies of a single element a called a generator of G.

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2. The identity matrix is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
3. The inverse of the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, which is in $\mathrm{GL}(2, \mathbb{R})$.

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*Note that this group is not commutative.

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The set $\mathrm{Mat}_2(\mathbb{R})$ is not a group under multiplication

The set is not a group under multiplication because not every matrix has an inverse. For example, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ does not have a multiplicative inverse because the determinant is 0.

Nonexample 2

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Integers under multiplication $(\mathbb{Z}, \times)$ are not a group

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This set is not a group because the inverse does not exist. For instance, there is no inverse of 2 since $1/2$ is not a integer.

Generators and Relations

Free Group

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Definition

The free group on elements $\langle x_1, x_2, ..., x_n \rangle$ consists of all finite-length words formed by $x_1, x_2, ..., x_n, x_1^{-1}, x_2^{-1}, ..., x_n^{-1}$.

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The free group on two elements was discussed in Example 3.

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Consider the free group on n elements, $x_1, x_2, ..., x_n$. Let $r_1, r_2, ..., r_m$ be elements in this group (these are just words). The group

$$\langle x_1, x_2, ..., x_n \mid r_1, r_2, ..., r_m \rangle$$

is the quotient we get by setting each r_i equal to identity.

Presentation of a Group

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Definition

A presentation of a group, G, is an expression of G in terms of generators and relations (shown in previous slide).

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The x represents the element 1, so $x^3 = e$ just means that $1 + 1 + 1 = 0 \pmod 3$.

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The x and y represent elements $(1, 0)$ and $(0, 1)$, and the relation $xy = yx$ just means that x and y commute.

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The symmetric group S_4 has presentation $\langle x_1, x_2, x_3 \mid x_1^2 = x_2^2 = x_3^2 = (x_1x_2)^3 = (x_2x_3)^3 = e \rangle$.

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The x_i represents the transpositions $(i, i+1)$.

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This is really large to work with by hand, so our examples have much nicer presentations!

Fun problems

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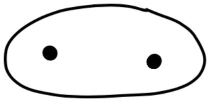

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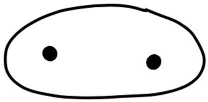

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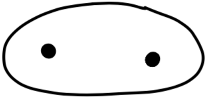

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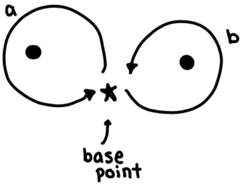

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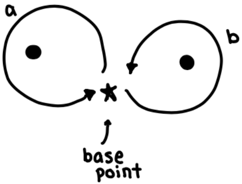

Loops (beginning and ending at the base point, up to homotopy) form a group by concatenation, with inverse being the reverse direction of the loop. This is called the fundamental group: in this case, the group is $\langle a, b \rangle$.

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Removing the left pole is the same as setting a to be the identity element. Similarly, removing the right pole is the same as setting b to be the identity element.
We must find an element $x \in \langle a, b \rangle$ that is not identity, but when either a or b is set to identity, x becomes the identity.

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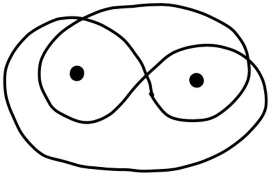

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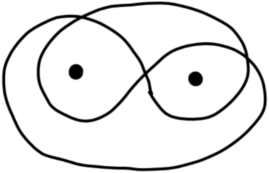

Generalization: If instead of 2 poles, there are n poles, can you find a loop which cannot be disentangled, but once any of the n poles are removed, then the loop can be removed?

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Let $a_1, a_2, \ldots, a_n$ be the generators of the fundamental group, where a_i is the counterclockwise loop around the ith pole.

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Let x_{n-1} be the solution representing $n-1$ poles. The element $x_{n-1} a_n x_{n-1}^{-1} a_n^{-1}$ represents the solution for n poles.

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Let x_{n-1} be the solution representing $n-1$ poles. The element $x_{n-1}a_nx_{n-1}^{-1}a_n^{-1}$ represents the solution for n poles. Why? When either one of the poles from 1 to $n-1$ are removed, x_{n-1} becomes the identity and the element becomes $a_na_n^{-1}$, which is identity. If the nth pole is removed, the element becomes $x_{n-1}x_{n-1}^{-1}$, which is also identity.

The Alphabet group

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Let's consider the free group generated by 26 generators, say a,b,c,d,...,x,y,z. Now impose the relations of homophones: that is, for every pair of words which are homophones (i.e. read and red), set them equal (i.e., read=red, where the generators are being multiplied). What is this group?

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Answer: There are many ways to arrive at the same answer. Here is one plausible solution.

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(5) $whole = hole \implies w = 1$

(6) $hour = our \implies h = 1$

(7) $in = inn \implies n = 1$

(8) $knot = not \implies k = 1$

(9) $die = dye \implies y = 1$

(10) $ad = add \implies d = 1$

(11) $all = awl \implies l = 1$

(12) $arc = ark \implies c = 1$

Possible Solutions continued...

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(13) $ate = eight \implies g = 1$

(14) $base = bass \implies s = 1$

(15) $berry = bury \implies r = 1$

(16) $boos = booze \implies s = 1$

(17) $bat = batt \implies t = 1$

(18) $check = cheque \implies q = 1$

(19) $idle = idol \implies o = 1$

(20) $lam = lamb \implies b = 1$

(21) $coo = coup \implies p = 1$

(22) $faze = phase \implies f = 1$

(23) $genes = jeans \implies j = 1$

(24) $flex = flecks \implies x = 1$

(25) $gamma = gama \implies m = 1$

Explanation of Solution:

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All letters except v are identity. Merriam-Webster finds that there are also no relations in v, so it turns out the quotient group is just $\langle v \rangle \cong \mathbb{Z}$.

Acknowledgements

We would like thank the MIT Math Department, MIT PRIMES, our family, you the audience, and Merrick Cai, our amazing mentor.