On the Distance Spectra of Extended Double Stars

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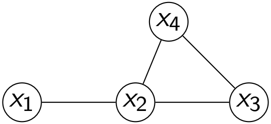
What is a graph?

Definition (Graph)

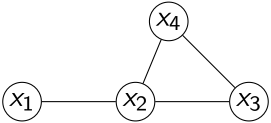
A (simple) graph has a vertex set V and an edge set E, where an edge is an unordered pair of distinct vertices of V.



Adjacency Matrices

Definition (Adjacency Matrix)

The adjacency matrix of a graph with vertices $x_1, x_2, ...x_n$ is the n by n matrix A where A_{ij} is equal to 0 if x_i and x_j don't have an edge connecting them, and equal to 1 if they do.

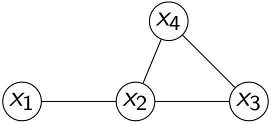


$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Distance Matrix

Definition (Distance Matrix)

The distance matrix of a graph with vertices $x_1, x_2, ...x_n$ is the n by n matrix D where D_{ij} is equal to the smallest number of edges that need to be traversed to get from x_i to x_j.



$$\begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix}$$

Matrix Multiplication

General Formula for Multiplying a Matrix by a Vector

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11}y_1 + a_{12}y_2 \ldots + a_{1n}y_n \\ a_{21}y_1 + a_{22}y_2 \ldots + a_{2n}y_n \\ \vdots \\ a_{m1}y_1 + a_{m2}y_2 \ldots + a_{mn}y_n \end{bmatrix}$$

Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \cdot 10 + 2 \cdot 11 + 3 \cdot 12 \\ 4 \cdot 10 + 5 \cdot 11 + 6 \cdot 12 \\ 7 \cdot 10 + 8 \cdot 11 + 9 \cdot 12 \end{bmatrix} = \begin{bmatrix} 68 \\ 167 \\ 276 \end{bmatrix}$$

Eigenvalues and Spectra

Definition (Eigenvalues)

An eigenvalue of an $n \times n$ matrix A is a scalar λ in which there exists some non-zero n by 1 vector v such that $Av = \lambda v$.

Definition (Spectrum)

The spectrum of a matrix is the "set" of its eigenvalues.

Example

$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Thus, we see that both 1 and 2 are eigenvalues of $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$.

Graphs Determined by their Spectra

Definition (Cospectral)

Two graphs are (distance) *cospectral* if they have the same (distance) spectrum. We refer to a graph as *determined by its (distance) spectrum* if it is not (distance) cospectral to any non-isomorphic graph.

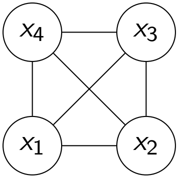
Existing Results

Definition (Diameter of a Graph)

The diameter of a graph is the maximum distance between any pair of vertices of the graph.

Existing Results

Existing results have focused on the adjacency spectrum. Diameter 2 graphs follow the pattern $D = 2J - 2I - A$, and most results on the distance spectra focus on low diameter.



Stars

Definition (Stars)

A *star* is a complete bipartite graph denoted by $K_{1,a}$, the graph formed by a single central vertex with a leaves connected to it.

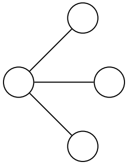


Figure: $K_{1,3}$

Double Stars

Definition (Double Stars)

An *double star*, denoted by $S(a, b)$, is the graph formed by joining the centers of the stars $K_{1,a}$ and $K_{1,b}$.

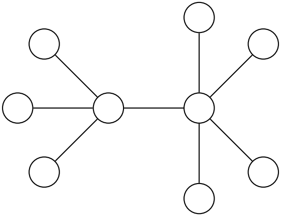


Figure: $S(3, 4)$

Extended Double Stars

Definition (Extended Double Stars)

An *extended double star*, denoted by $T(a,b)$, is the graph formed by joining the centers of the stars $K_{1,a}$ and $K_{1,b}$ to a common vertex.

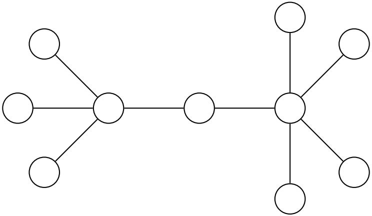
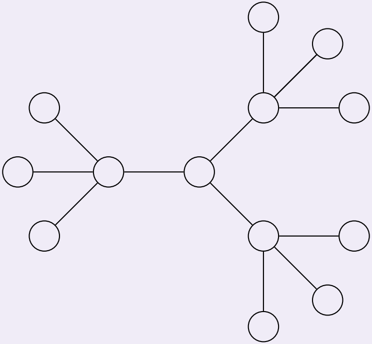


Figure: $T(3,4)$

Motivation

Triple Stars

We started with some diameter 4 trees, such as triple stars.



Definition (Induced subgraph)

For $S \subseteq V(G)$, an *induced subgraph* of G, denoted by $G[S]$, is the subgraph of G whose vertex set is S and whose edge set consists of all edges of G which have both ends in S.

Definition (Principal submatrix)

A principal submatrix is obtained by removing certain row indices and the same column indices.

Interlacing

Theorem (Interlacing)

Let G be a graph with n vertices and distance matrix $D(G)$. Denote its eigenvalues as $\lambda_1(D(G)) \geq \lambda_2(D(G)) \geq \ldots \geq \lambda_n(D(G))$. Let H be an induced subgraph of G with m vertices and distance spectrum $\mu_1(D(H)) \geq \mu_2(D(H)) \ldots \geq \mu_m(D(H))$. If $D(H)$ is a principal submatrix of $D(G)$, then $\lambda_{n-m+i}(D(G)) \leq \mu_i(D(H)) \leq \lambda_i(D(G))$ for $i = 1, 2, \ldots, m$.

Definition (Forbidden Subgraph)

A graph H is a *forbidden subgraph* of a graph G if the set of induced subgraphs of G does not include a graph isomorphic to H.

Our Results

Theorem

Extended double stars are determined by their distance spectrum.

Notation

The graph G is cospectral to an extended double star.

Diameter Not 3 or 4

Theorem (2013)

The complete graph K_n is determined by its distance spectrum.

Corollary (2016)

If G is a graph with order n and diameter 2, then $|E(G)| < |E(T)|$.

Theorem

If the diameter of G is greater than 4, then G and T are not cospectral.

Identifying a Path

There exist two vertices $x_1, x_5 \in V(G)$ such that $d_{x_1 x_5} = 4$. Denote $X = \{x_1, x_2, x_3, x_4, x_5\}$ the vertex set of a path of length 4.

Definition (Vertex sets V_i)

Denote by $V_i (i = 0, 1, 2, 3, 4, 5)$ the vertex subset of $V \backslash X$ consisting of vertices adjacent to i vertices of X.

Diameter 4 Graphs

Identifying a Path

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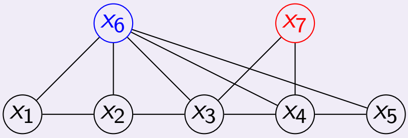
Example

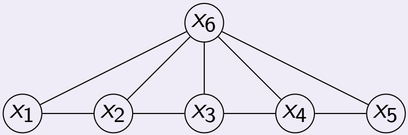


Here $x_6 \in V_5$ and $x_7 \in V_2$.

Diameter 4 Graphs

Forbidden Graphs

To prove V_5 empty, just show this subgraph is forbidden!



Diameter 3 Outline

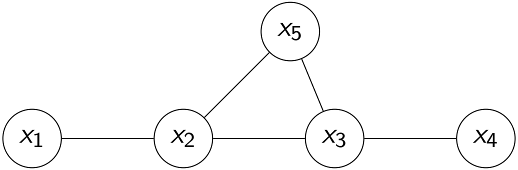
Proof Outline

- Prove V_4, V_3, V_1, V_0 are empty
- Prove V_2 is empty

Diameter 3 Outline

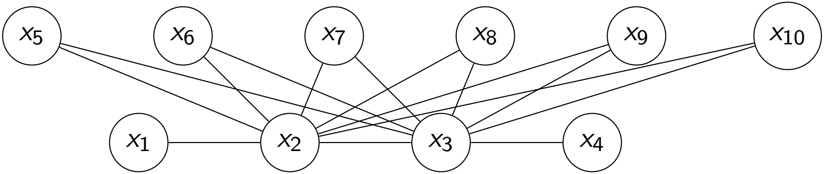
Definition

In V_2, interlacing eliminates all subgraphs except the one shown below. We call such a vertex in V_2 adjacent to x_2 and x_3 a *hat*.



Theorem

There can not be more than 5 hats in G.



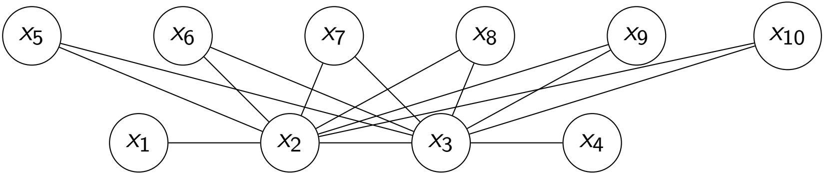
Theorem

There can not be 0 hats in G.

Diameter 3 Outline

Theorem

There can not be more than 5 hats in G.



Theorem

There can not be 0 hats in G.

Theorem

V_2 is empty.

Acknowledgements

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