A New Class of Atomic Monoid Algebras without the Ascending Chain Condition on Principal Ideals

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Monoid: definition

Monoid $(M, *)$ definition

For any $a, b \in M$, we can define $a * b \in M$, which satisfies $(a * b) * c = a * (b * c)$ (assoicativity).
Also, there exists $e \in M$ such that $e * a = a * e = a$ for any $a \in M$.

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Also, there exists $e \in M$ such that $e * a = a * e = a$ for any $a \in M$.

We will further assume that all monoids are commutative $(a * b = b * a)$ and cancellative ($a * c = b * c$ implies $a = b$).

Monoids: examples

Example: $(\mathbb{Z}_{\geq 0}, +)$

$\mathbb{Z}_{\geq 0} \triangleq \{n \in \mathbb{Z} : n \geq 0\} = \{0, 1, 2, 3, \ldots\}$ with operation $+$. We have $(1 + 2) + 4 = 3 + 4 = 7 = 1 + 6 = 1 + (2 + 4)$ and $0 + 5 = 5 + 0 = 5$. Similarly, $(\mathbb{Q}_{\geq 0}, +)$ is also a monoid.

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Example: $(\mathbb{Z} \setminus \{0\}, \cdot)$

$\mathbb{Z} \setminus \{0\} \triangleq \{n \in \mathbb{Z} : n \neq 0\}$ under multiplication. We have $(2 \cdot 3) \cdot 5 = 30 = 2 \cdot (3 \cdot 5)$. $1 \cdot 7 = 7 \cdot 1 = 7$.

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Remark

Note that $(\mathbb{Z}, +)$ is a monoid (since it is a group). However, we do not consider $(\mathbb{Z}, \cdot)$ because it is not cancellative ($0 \cdot 2 = 0 \cdot 3$ but $2 \neq 3$).

Monoid: generating monoid

A subset $N \subset M$ is called a submonoid if it is itself a monoid.
For any subset $S \subset M$, we let $\langle S \rangle$ denote the smallest submonoid in M containing S, more precisely,
$\langle S \rangle = \{n_1 s_1 + n_2 s_2 + \cdots + n_k s_k : n_1, n_2, \ldots, n_k \in \mathbb{Z}_{\geq 0}, s_1, s_2, \ldots, s_k \in S\}$.

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Example: a submonoid of $(\mathbb{Q}_{\geq 0}, +)$

$M = \langle \frac{1}{p} : p \text{ is prime} \rangle = \{\frac{n_1}{2} + \frac{n_2}{3} + \frac{n_3}{5} + \cdots + \frac{n_k}{p_k} : n_i \in \mathbb{Z}_{\geq 0}\}$.
$\frac{7}{6} = \frac{1}{2} + \frac{2}{3} \in M$ but $\frac{1}{6} \notin M$, $\frac{99}{25} \notin M$.

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Atomicity and ACCP: definitions

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Full name of ACCP

Ascending **C**hain **C**ondition on **P**rincipal ideals.

Atomicity and ACCP: examples

Example: $(\mathbb{Z} \setminus \{0\}, \cdot)$

$\mathbb{Z} \setminus \{0\}$ is atomic by the fundamental theorem of arithmetic. Furthermore, it satisfies the ACCP. For instance, the sequence $(64) \subset (32) \subset (16) \subset (8) \subset (4) \subset (2) \subset (1) \subset (1) \subset (1) \subset \cdots$ remains constant.

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A fun exercise

Prove that $M = \left\langle \frac{1}{p} : p \text{ is prime} \right\rangle \subset \mathbb{Q}_{\geq 0}$ (under the $+$ operation) satisfies the ACCP.

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Atomic v.s. ACCP

One can show if M satisfies the ACCP, then M is atomic. Therefore, we may ask a natural question: is there an atomic monoid that does not satisfy the ACCP?

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Example: Grams' monoid

$M = \left\langle \frac{1}{2^k p_k} : k \in \mathbb{Z}^+ \right\rangle$ (under the $+$ operation), where p_k is the k^{th} odd prime.

M is atomic: follows from the fact that each $\frac{1}{2^k p_k}$ is irreducible. Roughly speaking, this is because in the denominator of $\frac{1}{2^k p_k}$, the factor p_k is not shared by other generators.

M does not satisfy the ACCP: because the sequence of principal ideals

$$\frac{1}{2} + M \subset \frac{1}{2^2} + M \subset \frac{1}{2^3} + M \subset \cdots$$

in M $\left(\frac{1}{2^k} = p_k \cdot \frac{1}{2^k p_k} \in M \right)$ does not become constant.

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Integral domain

A ring R is an additive abelian group with a multiplication operation "$\cdot$" that distributes with "$+$." It is an integral domain if $(R \setminus \{0\}, \cdot)$ is a (cancellative) monoid, that is, $ab = 0$ implies $a = 0$ or $b = 0$.

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Example: integers

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The set of integers $\mathbb{Z}$ is an integral domain (under the $+, \cdot$ operations).

Atomicity and ACCP

An integral domain R is atomic (resp. satisfies the ACCP) if the multiplicative monoid $(R \setminus \{0\}, \cdot)$ is atomic (resp. satisfies the ACCP). For example, $\mathbb{Z}$ satisfies the ACCP.

Monoid algebras

Let $(M,+)$ be a monoid and R be a ring. Then ring $R[M] \triangleq \mathsf{Span}_R(x^m : m \in M) = \{\sum_{i=1}^{\ell} c_i x^{m_i} : c_i \in R, m_i \in M\}$, where the multiplication is defined by $x^{m_1} \cdot x^{m_2} = x^{m_1+m_2}$.

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Monoid algebra v.s. integral domain

Suppose R is an integral domain. When M is a submonoid of $(\mathbb{R}_{\geq 0}, +)$, $R[M]$ is also an integral domain.

Atomic v.s. ACCP for integral domains

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This is actually a pretty hard question. It was even asserted by Cohn back in 1968 that every atomic integral domain satisfies the ACCP. However, this assertion was disproved by Grams, who found an atomic domain that does not satisfy the ACCP using the monoid

$$M = \left\langle \frac{1}{2^k p_k} : k \in \mathbb{Z}^+ \right\rangle.$$

For any field F, he shows that the algebra $F[M]$ localized at $\{f \in F[M] : f(0) \neq 0\}$ is atomic without the ACCP.

Atomic Monoid Algebras without the ACCP

Such examples are elusive. In fact, there are only 4 papers describing such integral domains. In our research, we found a new class of integral domains that do not satisfy the ACCP. Furthermore, they are monoid algebras, which is presumably one of the most elementary structures.

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There exists a monoid algebra that does not satisfy the ACCP.

Construction:

1. Let $\{1, \alpha_n : n \in \mathbb{Z}^+\} \subset \mathbb{R}_{>0}$ be linearly independent over $\mathbb{Q}$ with $\sum_{n=1}^{\infty} \alpha_n < \frac{1}{4}$. Now set $A \triangleq \left\{\alpha_{j_k} + \sum_{i=1}^{\ell} \alpha_{j_i} : 1 \leq k \leq \ell \text{ and } 1 \leq j_1 < \cdots < j_\ell\right\}$.

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2. Set $B \triangleq \left\{ \beta_n \triangleq 1 - \sum_{i=1}^{\ell} \alpha_i : \ell \in \mathbb{Z}^+ \right\}$ and monoid $M \triangleq \langle A \cup B \rangle$.

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3. Fix a field F and consider $F[M]$. We showed that the monoid algebra $F[M]$ is atomic without the ACCP.

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