Counting LU Matrices with Fixed Eigenvalues

PRIMES conference 2021

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Example

Some common examples are $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$.

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When q is any prime p, $\mathbb{F}_q$ is like working modulo p. For any other q, it is slightly more complicated.

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Example

A common example is $\mathbb{Z}$ under addition.

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- Group under matrix multiplication
- Main group we will work with

Eigenvalues

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An **eigenvalue** for a matrix $g \in GL_n(\mathbb{F}_q)$ is any scalar λ for which there exists a vector $v \in \mathbb{F}_q^n$ such that $gv = \lambda v$. v is called an *eigenvector*.

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The eigenvalues of $\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$ are 2 and 4 with eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, respectively.

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$$\begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 4 & 14 \end{pmatrix}$$

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This is a well known fact.

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We call this set of matrices X_s.

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Case $n = 2$: LU matrices have the form

$$\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} ac & d \\ 0 & c^{-1} \end{pmatrix} = \begin{pmatrix} ac & d \\ abc & bd + c^{-1} \end{pmatrix}.$$

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The eigenvalues are roots of the polynomial
$t^2 - (ac + c^{-1} + bd)t + a$.
If $s = \{ae, e^{-1}\}$, we want roots to be ae and e^{-1}. Casework gives us $q^2 + 1$ solutions.

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A *computer program* seems more appropriate now. That gives $q^6 + 4q^3 + 1$.

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Is there a pattern? *Yes, but the formula is quite complicated.*

The Formula

Theorem (G)

For all n-element subsets s of $\mathbb{F}_q$, *we have*

$$|X_s| = \sum_{\lambda \in Y_n} q^{c(1)+c(\lambda)} D_\lambda^2.$$

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We will not talk about the proof because it is rather lengthy and complex.

Partitions

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A **partition** of a positive integer n is a way to write it as a sum of unordered positive integers. We can write a partition λ as $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$.

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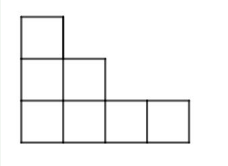

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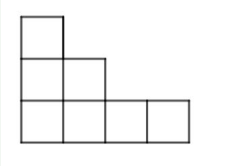

λ is $(4, 2, 1)$ in this example.

The Hook Length Formula

Definition

For (i, j) the square in row i, column j, we let $h(i, j)$ denote the number of squares (i', j') in the Young diagram λ such that $i' \geq i, j' = j$ or $i' = i, j' \geq j$. Then, the **Hook Length Formula** says $D_\lambda = \dfrac{n!}{\prod_{i,j} h(i, j)}$.

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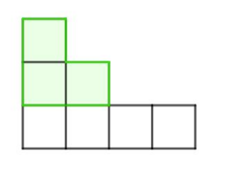

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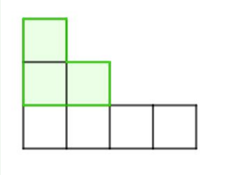

D_λ is $\frac{7!}{6 \cdot 4 \cdot 2 \cdot 1 \cdot 3 \cdot 1 \cdot 1} = 35$ in this example.

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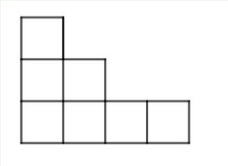

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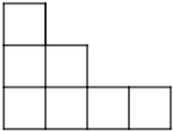

The content is 3 in this example.

Using The Formula

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After some work, we find for $n = 4$:

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After some work, we find for $n = 4$:

$$q^{12} + 3q^{10} + 2q^6 + 3q^2 + 1.$$

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