MIT PRIMES STEP JUNIOR GROUP

FUN WITH LATIN SQUARES

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1	2	3	4
4	3	2	1
3	4	1	2
2	1	4	3

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INTRODUCTION

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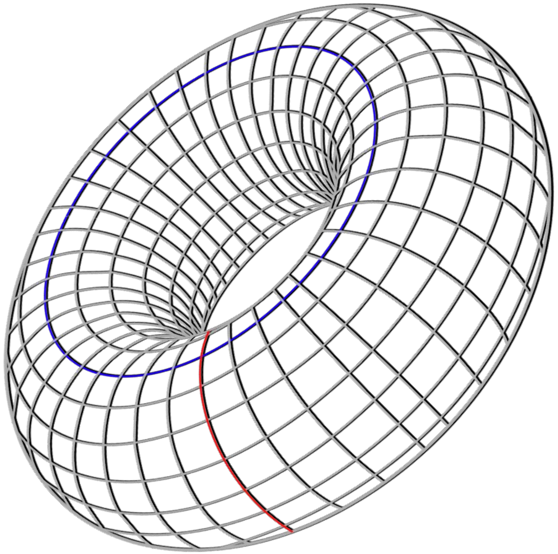
LATIN SQUARE

- *n* by *n* grid.
- 1 through *n* occur exactly once per row and column.

1	2	3
2	3	1
3	1	2

Toroidal Latin Squares

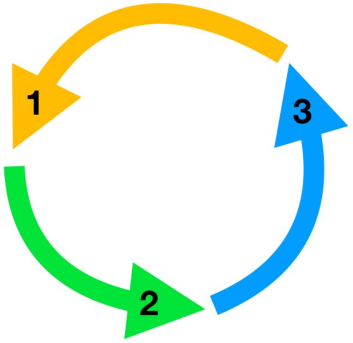
- Latin square inscribed onto torus.
- nth row is adjacent to the first row.
- nth column is adjacent to the first column.



Cyclic Latin Squares

- Every row cycled to left/right of previous row.
- Right-cyclic = cycles to the right; left-cyclic = cycles to the left.

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4



Special Latin Squares

WHAT IS A CHIECE?

We define “Chiece” to refer to any chess piece. It is a portmanteau of the two words chess and piece.

CHIECE LATIN SQUARES

- Every number is a chiece move away from another identical number.


dreamstime

1	2	3	4	5
5	4	1	3	2
4	3	2	5	1
2	5	4	1	3
3	1	5	2	4

Knight Latin Square of Size 5

CHIECE LATIN SQUARES (CONTD.)

- Bishop squares of order 5 do not exist.
- King Latin squares of odd sizes do not exist.
- Bishop Latin squares are equivalent to queen Latin squares.
- There exists a bishop Latin square for any even size.

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

CHIECE LATIN SQUARES (CONTD.)

Theorem 18. *If there exists an n by n chiece Latin square, then for all positive integers nm, there exists an nm by nm chiece Latin square.*

Proof. Consider a chiece Latin square N of order n and any Latin square M of order m. Multiplying N by M, another Latin square of order nm is obtained. This Latin square consists of m^2 blocks of chiece Latin square N, where each block is incremented by $n(m_{x,y}-1)$, where $m_{x,y}$ is an entry in M. Within its own block, each number has a copy of itself a chiece move apart, since the Latin square N is a chiece Latin square. Therefore, the Latin square of order nm must be a chiece Latin Square.

CHIECE LATIN SQUARES (CONTD.)

Here is an example using the theorem on the previous slide

1	2	3	4	5
5	4	1	3	2
4	3	2	5	1
2	5	4	1	3
3	1	5	2	4

1	2
2	1

CHIECE LATIN SQUARES (CONTD.)

1	2	3	4	5	6	7	8	9	10
5	4	1	3	2	10	9	6	8	7
4	3	2	5	1	9	8	7	10	6
2	5	4	1	3	7	10	9	6	8
3	1	5	2	4	8	6	10	7	9
6	7	8	9	10	1	2	3	4	5
10	9	6	8	7	5	4	1	3	2
9	8	7	10	6	4	3	2	5	1
7	10	9	6	8	2	5	4	1	3
8	6	10	7	9	3	1	5	2	4

ANTI-CHIECE LATIN SQUARES

- No number is ever a Chiece move away from another identical number.

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

Anti-Knight Latin Square of Size 5

Anti-Chiece Latin Squares (contd.)

- There are a total of 96 anti-knight squares of size 4, and 240 of size 5.
- Anti-bishop squares are equivalent to anti-queen squares.
- All Latin squares are anti-rook Latin squares.
- An anti-king Latin square exists for all *n*>6, where *n* is composite.
- Anti-queen Latin squares exist for all sizes not divisible by 2 or 3.

1	2	3	4	5
4	5	1	2	3
2	3	4	5	1
5	1	2	3	4
3	4	5	1	2

PROOF FOR LARGE ANTI-QUEEN SQUARES

Theorem 10. *We can construct an anti-queen square of size n by shifting the first row by k, where k, k+1, and k-1 are all coprime with n.*

- *Proof.* To make sure that none of the columns have multiple of the same number, *k* must be coprime with *n*.
- The diagonals going downwards from the left and right are shifted by *k*+1 and *k*-1 respectively, so to make sure that none of these repeat both *k*+1 and *k*-1 must also be coprime with *n*.

1	2	3	4	5
4	5	1	2	3
2	3	4	5	1
5	1	2	3	4
3	4	5	1	2

NOSY LATIN SQUARES

- Also known as consecutive Latin square.
- Two cells that share a side must contain consecutive digits.
- All consecutive Latin squares are also toroidal.
- In a modular square, 1 and *n* are considered consecutive.

1.	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

NOSY LATIN SQUARES (CONTD.)

- All Latin squares of size 1 & 2 are consecutive.

1

1	2
2	1

2	1
1	2

- There do not exist any non-modular consecutive Latin squares of size n, where $n>2$.
- Interestingly, for $n=4$, there exist non-cyclic modular nosy Latin squares.
- For $n>4$, there exist $4n$ nosy modular Latin squares.
- A modular consecutive Latin square with size $n>4$ is either left-cyclic or right-cyclic.

Shy Latin Squares

- Also known as non-consecutive Latin square.
- No number has an identical number orthogonally adjacent to it.

1	3	5	2	4
3	5	2	4	1
5	2	4	1	3
2	4	1	3	5
4	1	3	5	2

Shy Latin Squares (contd.)

- Shy Latin squares of size 5 are both anti-knight Latin squares and toroidal.
- Among all Latin squares of order 5, the lexicographically first is also anti-knight.
- Here is a non-toroidal shy Latin square:

1	3	5	2	4	6
3	5	1	4	6	2
5	1	3	6	2	4
2	4	6	1	3	5
4	6	2	3	5	1
6	2	4	5	1	3

→

1	3	5	2	4	6
3	5	1	4	6	2
5	1	3	6	2	4
2	4	6	1	3	5
4	6	2	3	5	1
6	2	4	5	1	3

Acknowledgements

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Our Family and Friends,

Especially our Parents.

THANK YOU for watching

Any Questions?

Works Consulted

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IMAGE CREDITS

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