Value sharing of meromorphic functions

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Fundamental Theorem of Algebra

Notation: $\mathbb{C}$ is the set of complex numbers, i.e., numbers of the form $a + bi$, where $i^2 = -1$ and a, b are real numbers.

Theorem: *Every degree-n polynomial $p(x)$ over $\mathbb{C}$, the field of complex numbers, has exactly n roots, counted with multiplicities.*

- Multiplicity of $p(x)$ at $c \in \mathbb{C}$: largest k such that $p(x)$ is divisible by $(x - c)^k$.
- This is not true for real numbers, since for example $x^2 + 1$ has no real roots, even though it has degree 2. However, $x^2 + 1$ has roots $i, -i$ in $\mathbb{C}$, each with multiplicity 1.

FTA restated: *If nonconstant complex polynomials $p(x)$ and $q(x)$ have the same preimages of 0 with the same multiplicities, then $p(x) = cq(x)$ for some constant c.*

This talk: generalize this to more complicated functions, and to preimages of *sets*, rather than points.

Shared multisets

Definition: A multiset is like a set, but where each element can occur multiple times.

Example: $\{1, 2, 2\}$ is a multiset of size 3.

- Write $|S|$ for the size of the multiset S.
- For any polynomial $p(x)$, write $p^{-1}(a)$ for the multiset of zeroes of $p(x) - a$. Thus $|p^{-1}(a)| = \deg(p)$.
- For a multiset S, write $p^{-1}(S)$ for the union $\bigcup_{a \in S} p^{-1}(a)$.
- If $p(x) = x^2$ we have $p^{-1}(\{0, 1, 2\}) = \{0, 0, 1, -1, \sqrt{2}, -\sqrt{2}\}$.
- Say polynomials p, q share a multiset S if $p^{-1}(S) = q^{-1}(S)$.

Characteristic polynomials

Definition: For a multiset S, the characteristic polynomial of S is

$$f_S(x) := \prod_{a \in S} (x - a).$$

Example: If $S = \{1, 2, 2\}$ then $f_S(x) = (x-1)(x-2)^2$.

A useful reformulation: p, q share $S \iff p^{-1}(S) = q^{-1}(S) \iff p^{-1}(f_S^{-1}(0)) = q^{-1}(f_S^{-1}(0)) \iff f_S \circ p$ and $f_S \circ q$ have the same roots, counting multiplicities.

Polynomials sharing multisets

Observation: *If nonconstant polynomials p, q share two disjoint nonempty finite multisets S, T then $g \circ p = g \circ q$ for some nonconstant polynomial $g(x)$.*

Proof: For $f(x) := \prod_{a \in S}(x - a)$, the roots of $f(p(x))$ are the p-preimages of S, counting multiplicities, which equal the roots of $f(q(x))$. So $f(p(x)) = cf(q(x))$, and then use T to show $c^n = 1$ for some $n > 0$, so that $f^n \circ p = f^n \circ q$. Q.E.D.

Remark: if $g \circ p = g \circ q$ for some nonconstant $g(x)$ then p, q share each of the *infinitely many* multisets $g^{-1}(a)$ with $a \in \mathbb{C}$, since

$$p^{-1}(g^{-1}(a)) = (g \circ p)^{-1}(a) = (g \circ q)^{-1}(a) = q^{-1}(g^{-1}(a)).$$

Rational functions sharing multisets

Definition: A rational function is one polynomial divided by another.

Definition: Functions p, q are quasi-equivalent if there exists a nonconstant rational function g such that $g \circ p = g \circ q$.

Observation: *If rational functions p, q share disjoint nonempty finite (multi)sets S_1, S_2, S_3 then they are quasi-equivalent.*

Remark: Quasi-equivalent p, q share infinitely many disjoint finite sets.

Meromorphic functions

Write $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ (the "Riemann sphere").

Meromorphic functions $p : \mathbb{C} \to \mathbb{C}_\infty$ are "well-behaved" functions, i.e., ratios of power series that converge everywhere on $\mathbb{C}$.

Example: Rational functions, trigonometric functions, and exponential functions are meromorphic. On the contrary, $|z|$ is not.

Theorem (Nevanlinna, 1926): *Meromorphic functions sharing* five points *are the same.*

Theorem (Nevanlinna, 1929): *If meromorphic* p, q *share* four points *then* $p = \mu(q)$ *for some degree-one rational function* $\mu(x)$.

Our main result generalizes these results to shared **(multi)sets**.

Meromorphic functions sharing sets

Main Theorem: *Let p, q be meromorphic functions sharing disjoint nonempty finite multisets $S_1, S_2, \ldots, S_n$, where $n \geq 4$. Then there is a rational function g such that $g \circ p = g \circ q$ and*

(1) $0 < \deg(g) \leq \frac{1}{n-3}(-2 + \sum_{i=1}^{n} |S_i|)$.

(2) *If $n \geq 5$ then* $0 < \deg(g) \leq \max_{i=1}^{n} |S_i|$.

- If such g exists then p, q share infinitely many sets of size $\deg(g)$.
- Four multisets is the best possible, since for example $p := (e^{x^2} - 1)/(e^x - 1)$ and $q := (e^{-x^2} - 1)/(e^{-x} - 1)$ share $\{0\}, \{1\}, \{\infty\}$ but are *not* quasi-equivalent.
- The bounds on the degree imply both of Nevanlinna's results.
- Meromorphic functions require more sets than rational functions because a function's zeroes and poles *don't* uniquely determine it up to a constant multiple. For example, e^x and 1 both have no zeroes or poles but are not constant multiples of each other.

Proof of Main Theorem

Lemma (Borel, 1897): *If $r_1, \ldots, r_k$ are meromorphic functions with no zeroes or poles, and $r_1 + \cdots + r_k = 0$, then for some $i \neq j$, r_i is a constant multiple of r_j.*

- To apply this lemma for p, q sharing $S_1, \ldots, S_n$, we must construct such $r_1, \ldots, r_k$ from p and q.

If p, q share S_i then $f_{S_i} \circ p$ and $f_{S_i} \circ q$ have the same zeroes, but possibly different poles.

Let $g_i = f_{S_i}^{|S_4|} / f_{S_4}^{|S_i|}$. Then $g_i(p)$ and $g_i(q)$ have the same zeroes and poles, so $g_i(p)/g_i(q)$ has no zeroes or poles.

Since we have three such functions g_1, g_2, g_3, there is a polynomial in the $g_i(x)/g_i(y)$ equaling 0, hence a polynomial in the $g_i(p)/g_i(q)$ equaling 0, where each term has no zeroes or poles. Thus the ratio of two terms is a constant c, yielding $g(p) = cg(q)$ for a rational function g. With more work we show $c^\ell = 1$ for some $\ell > 0$, so $g^\ell(p) = g^\ell(q)$.

Degree bounds

We have shown that if p, q share $S_1, \ldots, S_n$ with $n \geq 4$ then $g(p) = g(q)$ for some nonconstant rational function $g(x)$. Pick one such $g(x)$ of the smallest possible degree.

We show $\deg(g) \leq \frac{1}{n-3}(-2 + \sum_{i=1}^{n}|S_i|)$ via the Riemann-Hurwitz formula, the fact that any meromorphic parametrization of a singular curve must factor through its normalization, and the fact that there are no nonconstant holomorphic maps from $\mathbb{C}$ to a hyperbolic Riemann surface.

For $n \geq 5$ we show $\deg(g) \leq \max_{i=1}^{n}|S_i|$ by proving that one of the S_i's must contain a multiset of the form $g^{-1}(a)$, so that $\deg(g) \leq |S_i|$ for some i. This is hard.

Minimal shared multisets

Definition: A multiset S shared by p and q is minimal if p and q do not share any nonempty proper sub-multiset of S.

It's easy to show that shared multisets are precisely the unions of minimal shared multisets, so to determine the shared multisets it suffices to determine the minimal shared multisets:

Theorem: *If p, q are quasi-equivalent, and g is of minimal degree such that $g \circ p = g \circ q$, then all but at most four minimal shared multisets are of the form $g^{-1}(a)$.*

- The minimal shared multisets not of the form $g^{-1}(a)$ come from one of two sources: the "missed values" of p and q, or the possibility that some $\ell > 1$ divides all multiplicities in $g^{-1}(a)$, e.g., $(x^2)^{-1}(0) = \{0, 0\}$.
- Proof uses Galois theory, algebraic topology, and algebraic geometry.

Other problems

The same methods can be used for other situations:

Theorem: *If meromorphic functions p, q are such that there are* five *pairs of nonempty disjoint multisets (S_i, T_i) such that $p^{-1}(S_i) = q^{-1}(T_i)$, then there are rational functions g, h such that $g \circ p = h \circ q$.*

Theorem: *Rational functions on a smooth projective curve C (over an algebraically closed constant field) which share* three *nonempty disjoint multisets are quasi-equivalent.*

Theorem: *Meromorphic functions on a (complete, algebraically closed) non-archimedean field which share* three *nonempty disjoint multisets are quasi-equivalent.*

We have a similar bound on the degree of the algebraic relation in all of these cases, and characterize the minimal shared multisets.

Questions

- Can similar results be proved for sharing sets ignoring multiplicity?
- What can be said about meromorphic functions sharing fewer than 4 multisets?
- What other types of functions can the results be generalized to? meromorphic functions on complex manifolds? rational functions on varieties? on schemes??

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GCD of multiplicities in $g^{-1}(a)$

Theorem: *If meromorphic p, q are quasi-equivalent, and $g(x)$ is a minimal-degree rational function with $g(p) = g(q)$, then there are at most two points a for which the gcd of the multiplicities in $g^{-1}(a)$ is bigger than* 1.

Proof sketch: Show that if the gcd e_i of the multiplicities in $g^{-1}(a_i)$ satisfies $e_i > 1$ for $i = 1, 2, 3$ then $g = f(h)$ where $\deg(f) > 1$ and $\mathbb{C}(x)/\mathbb{C}(f(x))$ is Galois with non-cyclic group. Thus $f(x) - f(y)$ factors as $a(x) \cdot b(y) \cdot \prod_j (x - \mu_j(y))$ for some $a, b, \mu_j \in \mathbb{C}(x)$ with $\deg(\mu_j) = 1$. Hence $h(p) = \mu_j(h(q))$ for some j, so if $\mathbb{C}(u(x))$ is the subfield of $\mathbb{C}(x)$ fixed by the automorphism $\sigma_j \colon x \mapsto \mu_j(x)$ then $u(h(p)) = u(h(q))$. Since $G := \mathrm{Gal}(\mathbb{C}(x)/\mathbb{C}(f(x)))$ is non-cyclic, $\deg(u) = |\langle \sigma_j \rangle| < |G| = \deg(f)$, so $\deg(u(h)) < \deg(g)$, contradicting minimality of $\deg(g)$.

Algebraic topology

Suppose the gcd e_i of the multiplicities in $g^{-1}(a_i)$ satisfies $e_i > 1$ for $i = 1, 2, 3$. View g as a branched covering $S^2 \to S^2$. Klein (1886) constructed $f(x)$ with $\mathbb{C}(x)/\mathbb{C}(f(x))$ Galois but non-cyclic, where all multiplicities in $f^{-1}(a_i)$ equal e_i. Writing B for the set of branch points of g, the restrictions of g and f to $S^2 \setminus g^{-1}(B)$ and $S^2 \setminus f^{-1}(B)$ are topological covers ϕ and ψ, and any component X of the pullback of ϕ along ψ satisfies

$$\begin{array}{ccc} X & \xrightarrow{\pi_2} & S^2 \setminus f^{-1}(B) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \psi} \\ S^2 \setminus g^{-1}(B) & \xrightarrow{\phi} & S^2 \setminus B \end{array}$$

The compactification of π_1 yields an *unbranched* cover of S^2, which must be a homeomorphism since S^2 is simply connected, so $g = f \circ h$.

Proof of the degree bound

Theorem: *If meromorphic p, q share multisets $S_1, \ldots, S_n$ with $n \geq 4$ then* $\deg(g) \leq \frac{1}{n-3}(-2 + \sum_{i=1}^{n} |S_i|)$.

Here is a proof in the easiest case, when each S_i contains a minimal shared multiset of the form $g^{-1}(a)$:

Lemma (Riemann-Hurwitz, 1857): *A rational function h of degree ℓ satisfies*

$$2\ell - 2 = \sum_{a \in \mathbb{C}_\infty} (\ell - |h^{-1}(a)_{set}|),$$

where S_{set} is the underlying set of a multiset S.

Applying the theorem for g with degree k gives

$$2k - 2 \geq \sum_{i=1}^{n} \big(k - g^{-1}(a_i)\big) \geq nk - \sum_{i=1}^{n} |S_i|, \text{ so } k \leq \frac{1}{n-2}\Big(-2 + \sum_{i=1}^{n} |S_i|\Big).$$