K-semistability of Smooth Toric Fano Varieties

Alvin Chen
Mentor: Kai Huang

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(Algebraic) Varieties

We will work in projective n-space $\mathbb{P}^n$.

Definition

Classically, a **variety** is a set of points in space that all satisfy some number of polynomial equations. We can think about the points in space as n-dimensional coordinates $(x_1, x_2, \ldots, x_n)$.

Examples

- A line
- A surface
- The entire space
- A torus in n dimensions

Background

Kähler–Einstein (KE) metric

- Important type of Riemannian metric on complex manifolds
- Useful to differential geometers

Which manifolds have a KE metric?

The Minimal Model Program classified algebraic varieties into three types:

- General type
- Calabi-Yau
- Fano

K-stability

Fano

Fano case is most difficult: Due to the Yau-Tian-Donaldson Conjecture (proved 2015), existence of a KE metric is equivalent to K-stability, a condition in algebraic geometry.

α-invariant

We can use a numerical invariant assigned to varieties to investigate K-semistability.

Our Assumptions on Varieties

It's difficult to do computations with general varieties, so we need some restrictions on the varieties we use.

Restrictions

Our goal is to perform computations of $\alpha(X)$ for all varieties that are:

1. Projective
2. Smooth
3. Toric
4. Fano

Why Toric?

K-stability was originally introduced with toric varieties.

Toric test configurations

Much easier to do computations with toric varieties.

Fact

Every polytope corresponds to a projective toric variety.

Good News

The conditions that we would like to apply to varieties can be restated as combinatorial conditions, and $\alpha(P)$ can as well.

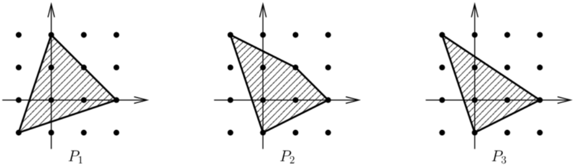
Polytopes

Definition

A **lattice polytope** P is the convex hull of a set of points in $\mathbb{Z}^n$. $\mathcal{V}(P)$ denotes the vertex set of P.

Definitions regarding polytopes

A **face** of a polytope P is the intersection of P with some hyperplane. A **vertex** is a zero-dimensional face, and a **facet** is a codimension one face.



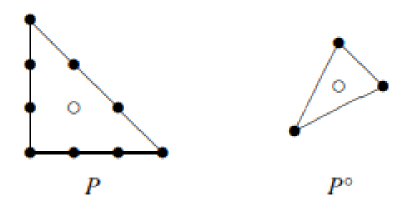
Dual Polytopes

Definition

The **dual polytope** P° to a polytope P is defined by

$$P^\circ = \{u \in \mathbb{Z}^n | \langle u, v \rangle \geq -1 \quad \forall v \in \mathcal{V}(P)\}.$$

Two-dimensional Example



Conditions on Polytopes

Fano

All polytopes corresponding to Fano varieties **reflexive**, which means that their dual is a lattice polytope as well.

Definition (Smooth Fano)

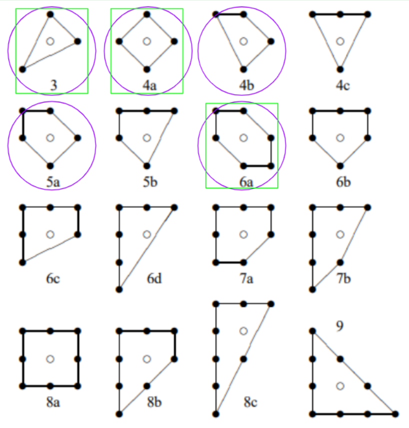
A polytope is smooth and Fano if and only if every facet is an integral basis for $\mathbb{Z}^n$.

Definition (K-semistability)

A polytope is **K-semistable** if the vector sum of the vertices is 0.

Example

Two dimensional Reflexive Polygons

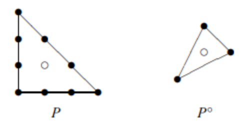


α-invariant with Polytopes

Definition

The α-invariant $\alpha(P)$ of a polytope P is min $\frac{1}{\langle u,v\rangle+1}$ over all (u,v), where $u \in \mathcal{V}(P)$, $v \in \mathcal{V}(P^\circ)$, and $\langle u,v\rangle$ is the dot product.

α of $\mathbb{P}^n$



The largest dot product is $(1,0)\cdot(2,-1)=2$, so $\alpha(P)=\frac{1}{3}$.

Gap Conjecture

Question

What values can $\alpha(X)$ obtain?

Conjecture (Jiang)

If X is a smooth Fano K-semistable variety and $\alpha(X) < \frac{1}{n}$, then $X \cong \mathbb{P}^n$.

Remark

This means $\alpha(X)$ has a gap, as $\alpha(\mathbb{P}^n) = \frac{1}{n+1}$.

Results

We proved the gap conjecture for $\alpha(X)$ in the toric case assuming a simplified version of Ewald's conjecture.

Conjecture (Ewald)

Every toric smooth Fano polytope is isomorphic to one contained in $[-1,1]^n$.

Theorem

The possible values of $\alpha(X)$ *for* X *as toric smooth Fano K-semistable variety are* $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n+1}$.

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References

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