Colored HOMFLY Polynomials of Genus-2 Pretzel Knots

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Knots

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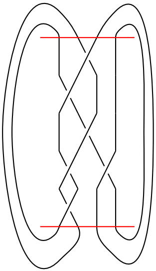

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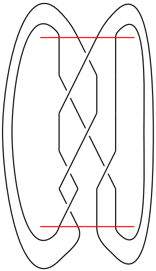

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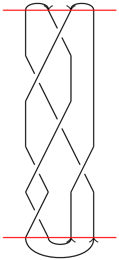


(b) One Branch of a Double-Fat Diagram

Figure: Different knot presentations

Knot Classification

Definition (Pretzel Knots)

A **pretzel knot** is a knot of the form shown. The parameters used here are the numbers of twists in the ellipses. We can have many ellipses.

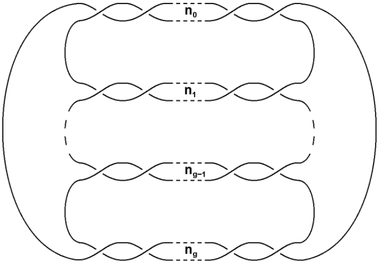
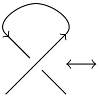


Figure: An illustration of a genus g pretzel knot

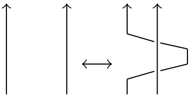
Reidemeister Moves

Proposition (Reidemeister Moves)

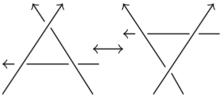
If and only if a sequence of Reidemeister moves can transform one knot or link to another, they are equivalent. The three Reidemeister moves are illustrated below.



(a) Move 1



(b) Move 2



(c) Move 3

Figure: The Reidemeister Moves

Knot Invariants

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A **knot invariant** is a quantity that doesn't change after the application of a Reidemeister move. It is therefore independent of projection.

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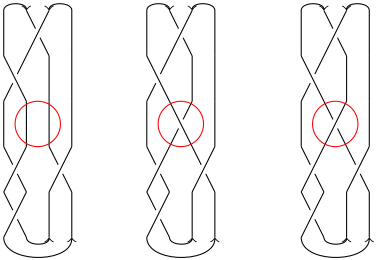
Definition (Knot Polynomial)

A **knot polynomial** is a type of knot invariant that is expressed as a polynomial.

Computing knot polynomials using skein relations

Definition (Skein relation)

A **skein relation** is a relationship between different kinds of intersection. With it, a knot polynomial can be computed recursively.



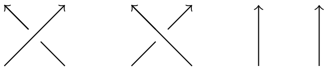
(a) L_0 (b) L_- (c) L_+

Figure: Skein relation computation

HOMFLY Polynomial

Definition (HOMFLY Polynomial)

The **HOMFLY polynomials** are knot polynomials defined by the skein relation $\frac{1}{A}\mathcal{H}(L_+) - A\mathcal{H}(L_-) = (q - q^{-1})\mathcal{H}(L_0)$, and $\mathcal{H}(unknot) = 1$.



(a) L_+ (b) L_- (c) L_0

Figure: The Crossing Types

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- Symmetric representations are [r], Anti-symmetric representations are [1,1,1,1,1,$\cdots$], fundamental representation is [1].

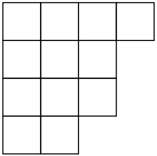

Figure: The Young diagram [4,4,3,1]

S and T matrices

Proposition (Conformal Field Theory Method)

Racah matrices come in two types: S and T. The T matrices are the crossing matrices, and S matrices bring strands closer or further from each other. Both types come in 4 forms.

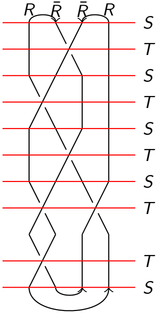

Figure: An example double bridge knot

Differential Expansion

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The **differential expansion** is an expression of HOMFLY polynomials as the sum of some "quantum numbers" $\{x\} = x - x^{-1}$ and $[x] = \frac{\{q^x\}}{\{q\}}$ and some polynomial "F-factors", allowing recursive expression of HOMFLY.

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Conjecture (Form Of Differential Expansion)

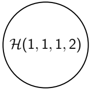
The HOMFLY polynomial of any defect-zero knot in a representation $[r]$ *can be expressed as*

$$\sum_{k=0}^{r} \frac{[r]!}{[k]![r-k]!} F_{[r]}(A,q) \prod_{i=0}^{k-1} \{Aq^{1-i}\}\{Aq^{-r-i}\}.$$

Differences

Definition (n^{th} differences)

The n^{th} **difference** of a genus 2 pretzel knot with fixed a, b, denoted $Q^n(c, r)$, is equal to the difference of (the largest polynomial factors) of the $n - 1^{st}$ differences. In particular, if the largest polynomial factor is not taken, the monomial factors are cleared.



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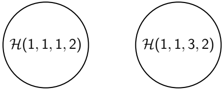

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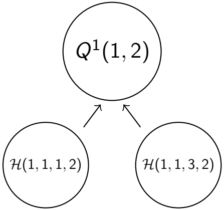

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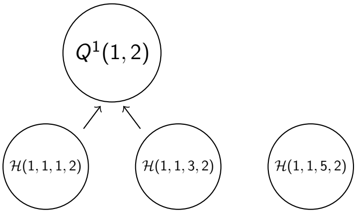

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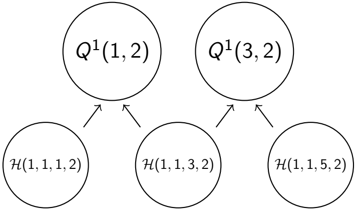

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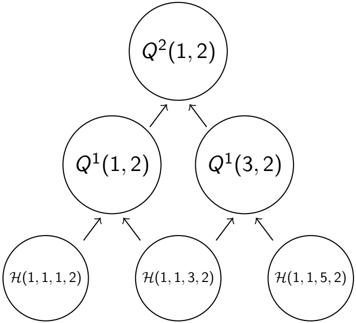

Figure: Difference Computations

Main Conjecture

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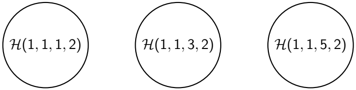
$Q^r(c,r) \approx Q^r(c+2,r)$ *up to* $A^r q^{2(r)(r-1)}$



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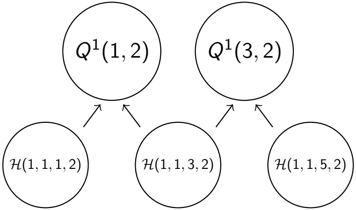

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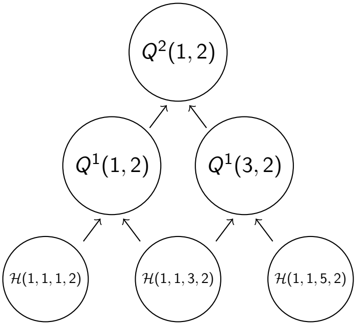

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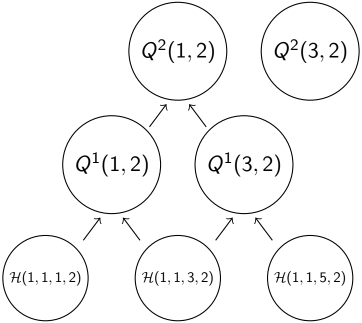

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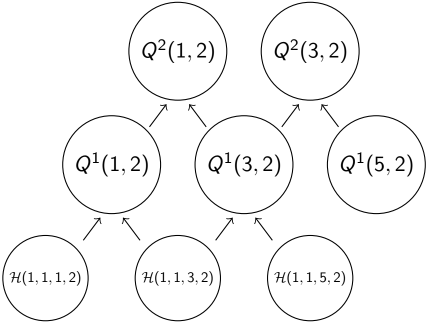

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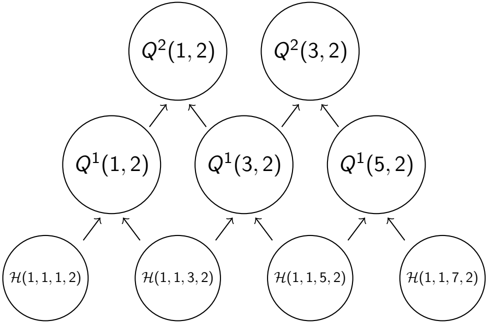

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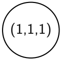
Implications

1. This allows us to speed up HOMFLY calculations!
2. It is also true (this is a small lemma) that $H(a, b, c, r) = H(a, c, b, r) = \cdots = H(c, b, a, r)$.



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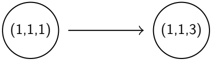

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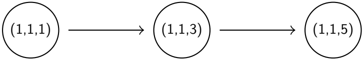

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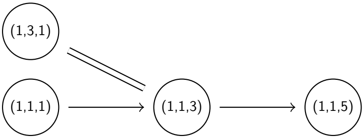

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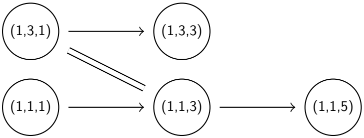

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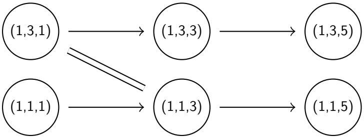

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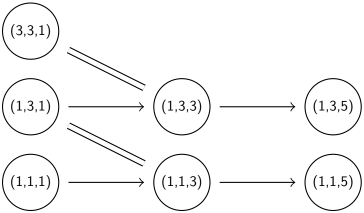

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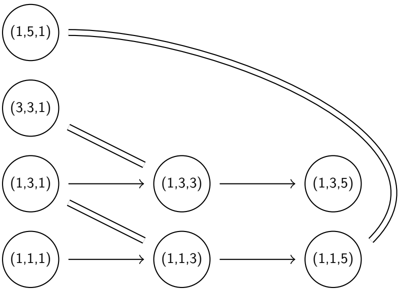

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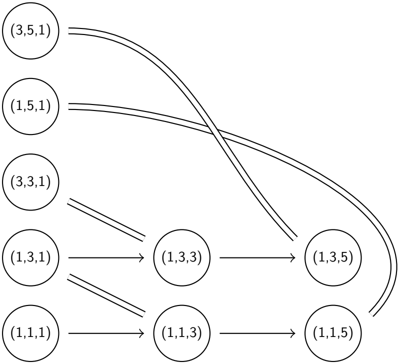

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3. $$\overline{S} = \begin{pmatrix} \frac{A(q^2-1)}{(A^2-1)q} & \frac{A(q^2-1)\sqrt{\frac{(A-q)(A+q)(Aq-1)(Aq+1)}{A^2(q^2-1)^2}}}{(A^2-1)q} \\ \frac{A(q^2-1)\sqrt{\frac{(A-q)(A+q)(Aq-1)(Aq+1)}{A^2(q^2-1)^2}}}{(A^2-1)q} & \frac{A(q^2-1)}{(A^2-1)q} \end{pmatrix}.$$

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3. Then, taking the difference, we get 0, meaning that $A - q$ is a factor.
4. This works for $A = -q$, $A = \frac{1}{q}$, and $A = -\frac{1}{q}$ as well.

Acknowledgements

1. Yakov Kononov, for mentoring me and proposing the problem
2. MIT PRIMES, for both this research opportunity and helping facilitate the research
3. My parents

References

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