Knot Theory

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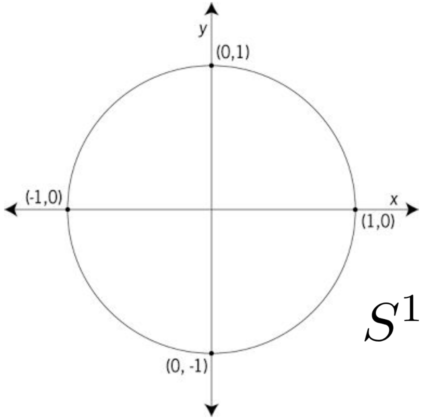
Overview

- Standard Definitions (Nithin)
- Knot Invariants (Nithin)
- Seifert Surfaces (Anmol)
- Genuses (Anmol)

What is a knot?

- Definition: A knot is a smooth embedding from S^1 to $\mathbb{R}^3$



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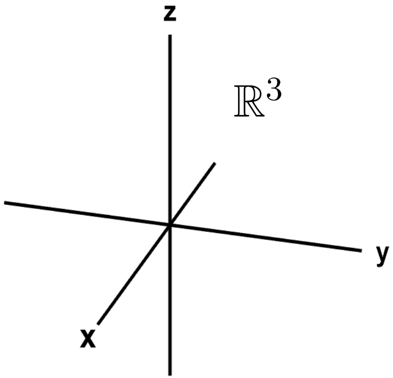
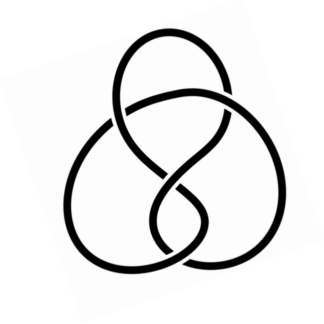


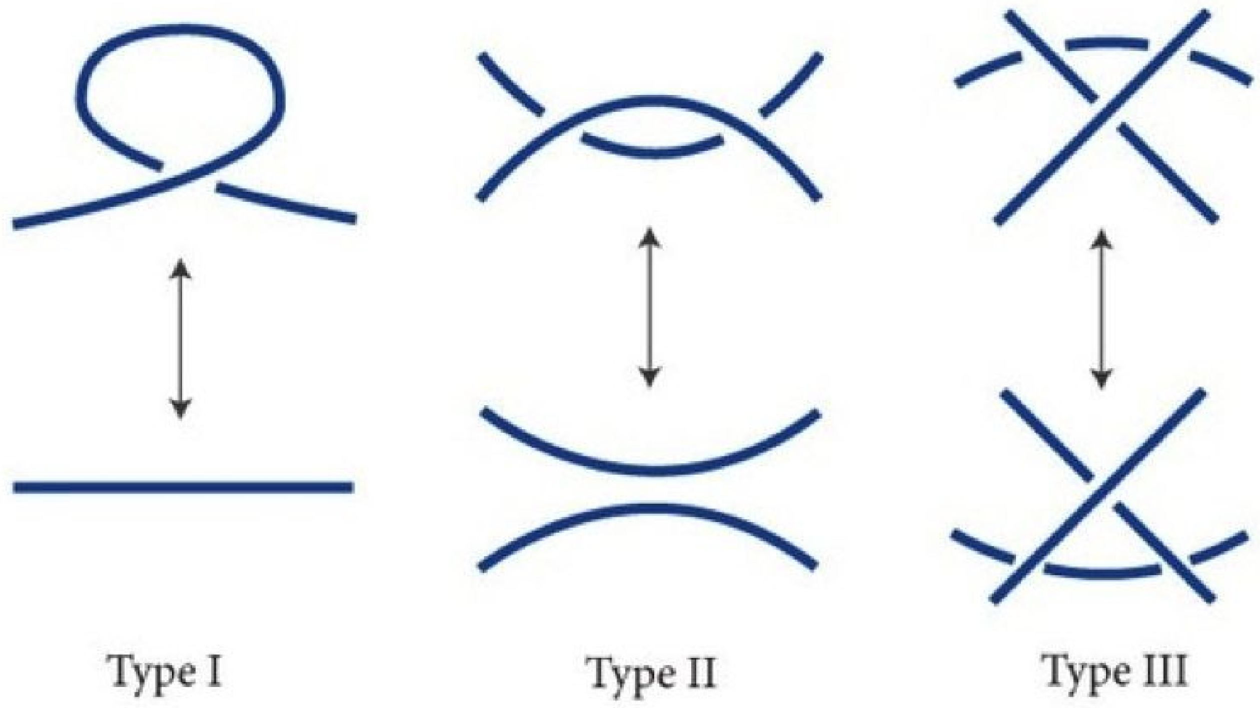
Result:

Important Definitions

- Definition of knot class: class of knots where any two knots can be deformed into each other in $\mathbb{R}^3$ without any self crossings during the deformation
- Definition of knot diagram: a projection of a knot k to a 2 dimensional plane in $\mathbb{R}^3$ such that only double crossings exist

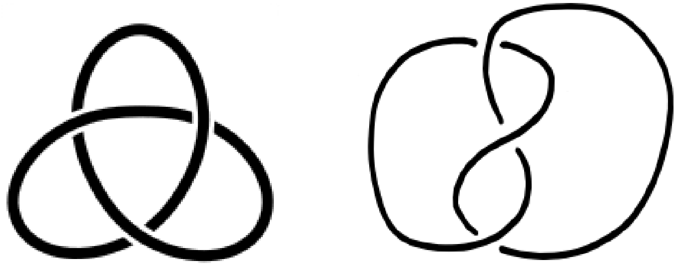


Reidemeister Moves



What is the significance of Reidemeister moves?

- Theorem (Reidemeister): If one knot diagram can be turned into another using any sequence of the Reidemeister moves, in combination with planar isotopies (deformation) then they belong to the same knot class
- Both of the following are actually the trefoil:



How can we describe knots?

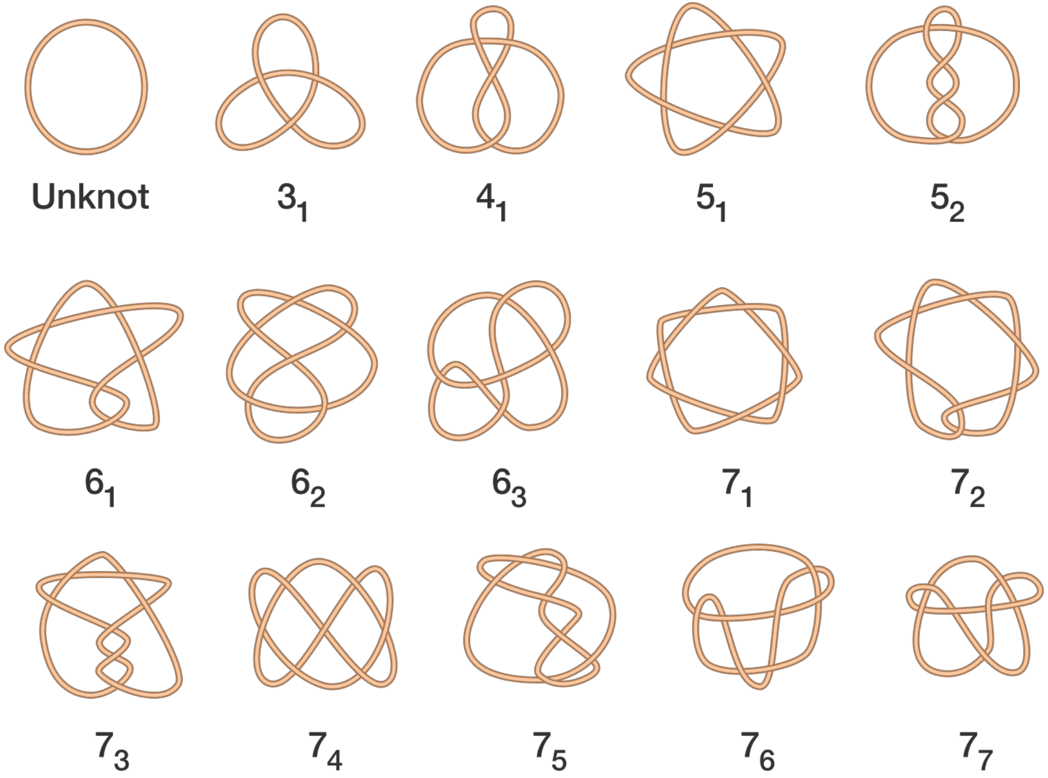
- Knots have invariants, including the crossing number, width number, trunk number, bridge number and more
- These invariants can be used to describe different knot classes
- Convention: a knot is generally denoted k, while a knot class is denoted K

Knot Invariants

Crossing Number

- Number of crossings of a knot
- The crossing number of a knot class is the minimum crossing number over all of the knots in the class
- Open conjecture: given a fixed positive integer c, the number of knots with crossing number c is approximately e^{ac+b} for some suitable constants a, b, for large enough c

Types of Knots



Local Maxima, Minima and Levels

- We define a standard height function h on $\mathbb{R}^3$ such that $h(x_1, x_2, x_3) \to x_3$.
- Under h, knots have local maxima and local minima, called critical points or critical levels; we generally only consider knots where all critical points are in different levels
- They are denoted $c_1, c_2, \ldots c_{n-1}, c_n$ from lowest to highest
- The regular levels are $r_1, r_2, \ldots r_{n-1}$ where $c_1 < r_1 < c_2 < r_2 < \ldots < c_{n-1} < r_{n-1} < c_n$.

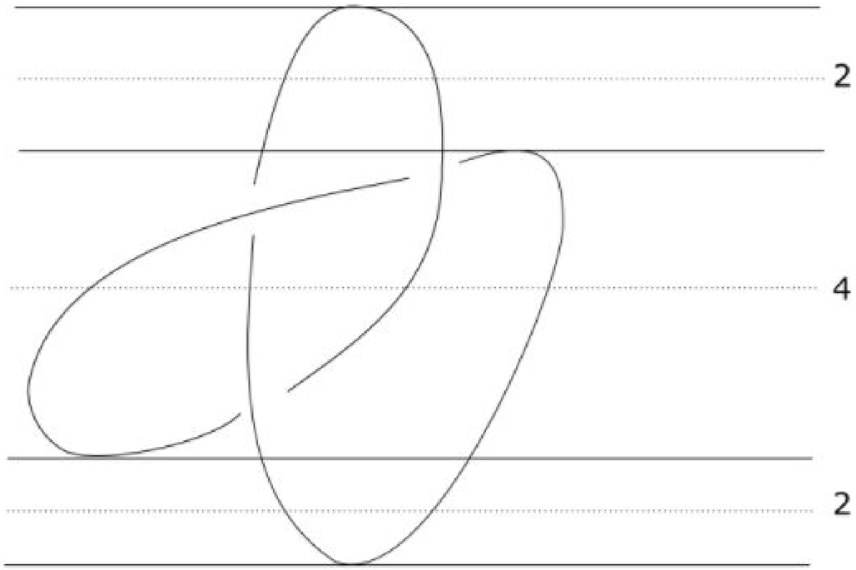
Bridge Number

- The number of local maxima of a knot
- The number of local minima of a knot
- Half the number of critical points
- b(K) is the minimum across all b(k) where k is a knot in K

Width and Trunk Number

- Count the intersections of each regular level with the knot
- The width of the knot is the total number of intersections
- w(K) is the minimum w(k) across all k in K
- The trunk of the knot is the maximum number of intersections across all of the regular levels
- tr(K) is the minimum tr(k) across all k in K

Image Illustrating Bridge, Trunk and Width Number



The trefoil has two local minima and two local maxima, so its bridge number is 2.

The regular levels intersect the trefoil 2, 4, and 2 times, so its width number is 2 + 4 + 2 = 8.

The maximum number of times any regular level intersects the the trefoil is 4, so its trunk number is 4.

Satellite and Companion Knots

- Consider a knot j, and a second knot k created by going around the knot j n times, following the shape of j
- Then k is the satellite knot with companion j
- The winding number of k is n
- Theorem (Li, Guo): $w(K) \geq n^2 \cdot w(J)$

Research

- Theorem: $tr(K) \geq n \cdot tr(J)$
- Difficulty: critical points of k must be described by critical points of j
- Solution: construct a loop l such that the critical points of k can be described b

  critical points of l
- Zupan: $tr(L) \geq tr(J)$

Seifert Surfaces

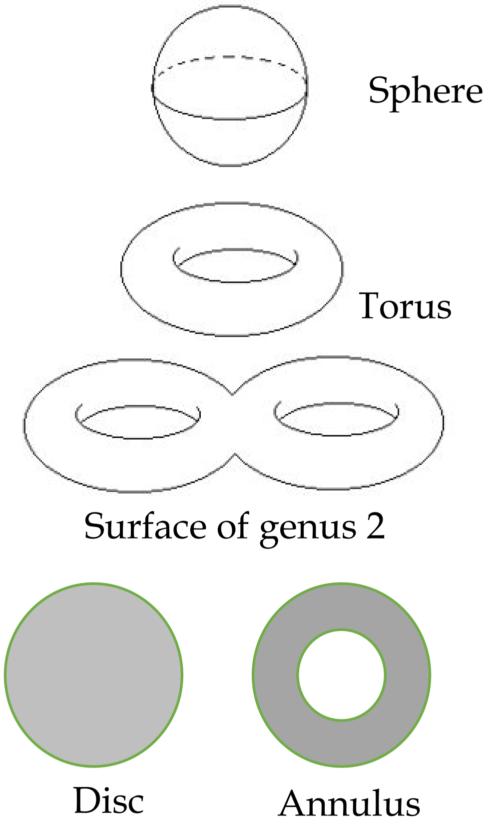
Definitions

Surface

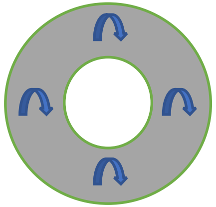
A Surface is a bounded two-dimensional object.

Closed

A surface is called closed if it has no boundary.
The sphere, torus and surface of genus 2 are all closed.
The disc and the annulus both have boundary



Connected/Disconnected



If a surface can be written as the union of two other surfaces that do not meet, then it is disconnected. Otherwise it is connected.

Orientable

An Orientation of a surface is a choice of twist at each point of the surface, and this choice must be consistent as you move around the surface. If a surface admits an orientation, it is orientable.

Euler Characteristics

Euler characteristic of surface Σ is

$$\chi(\Sigma) = v - e + f$$

If Σ is a closed surface of genus g, then

$$\chi(\Sigma) = 2 - 2 * g$$

For any surfaces Σ_1 and Σ_2

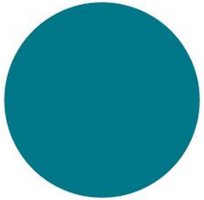
$$\chi(\Sigma_1 \# \Sigma_2) = \chi(\Sigma_1) + \chi(\Sigma_2) - 2$$

If Σ is obtained by attaching the endpoints of a strip to an existing surface Σ_1, then

$$\chi(\Sigma) = \chi(\Sigma_1) - 1$$

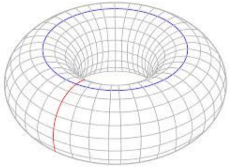
If Σ is obtained by attaching a disc to an entire boundary component of an existing surface Σ_1, then

$$\chi(\Sigma) = \chi(\Sigma_1) + 1$$



X(Disk) = 1

X(Sphere) = 2



X(Torus) = 0

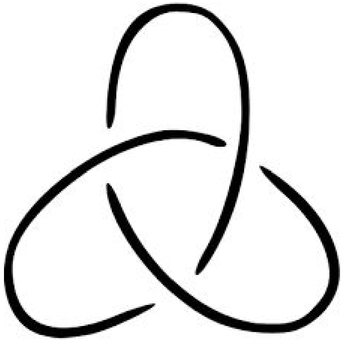
Seifert Surface

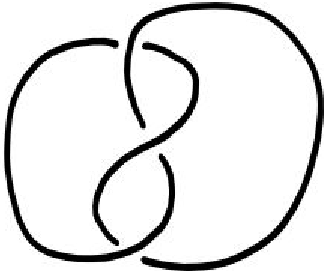
A Seifert surface for a knot K is a connected, orientable surface embedded in R^3 with boundary K

Any knot has a Seifert surface.

Depending on projections of the knot, Seifert algorithm may yield different surfaces for the same knot.

The surfaces formed from different projections may have different genus





Seifert Algorithm

1. Choose a diagram D of the knot.

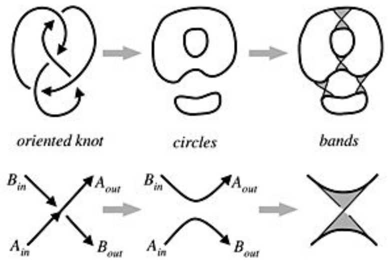
2. Give it an orientation.

3. Eliminate crossings: For each crossing two strands come in and two go out. Then connect each of the strands coming into the crossing to the adjacent strand leaving the crossing.

4. This results into a set of oriented circles on the plane. These circles are called *Seifert circles*

5. For each crossing of the original diagram, add a strip with a twist to the relevant discs, twisted according to the crossing of the knot.

The result is an oriented surface Σ_D



Genus

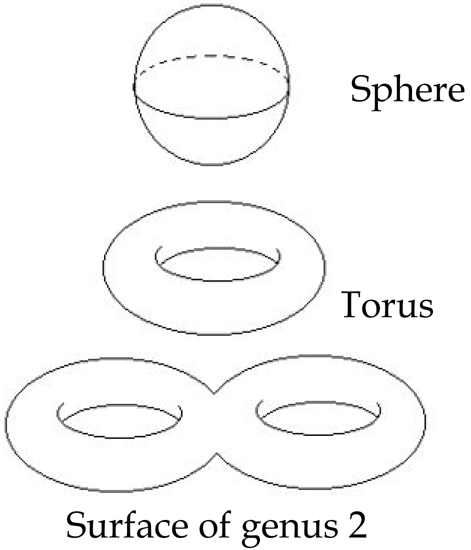
The genus of a compact surface is the number of "holes" it has.

The genus of a knot K, denoted as g(K) is the minimum genus of all Seifert surfaces for K

$$g(K) = min\{g(\Sigma)\}$$

Σ: Seifert surface for knot K

genus of sphere is 0, torus is 1, surface of genus 2 is 2



Let K be a knot and D be a diagram of K. Then

$$\chi(\Sigma_D) = s - n$$

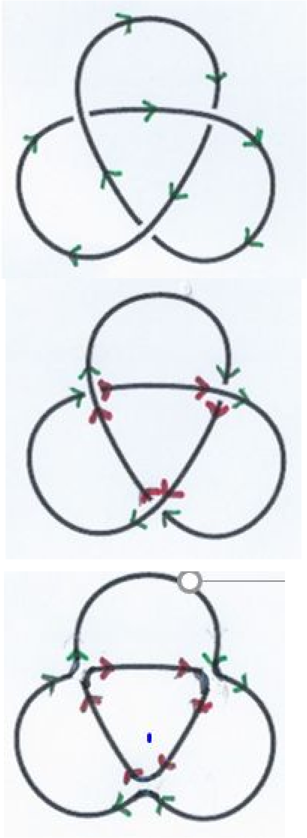
$$g(\Sigma_D) = (1 - s + n)/2$$

$$g(\Sigma_D) = (1 - \chi(\Sigma_D))/2$$

(n : # of crossings of D, s = # of Seifert circles)

Applying Seifert's algorithm to the trefoil:

n = 3, s = 2, genus of trefoil is : 1



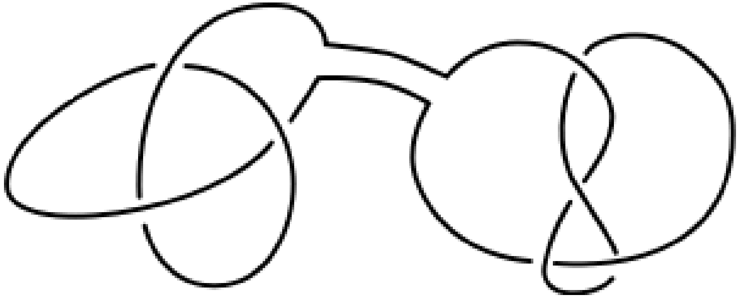
Connected Sum:

To form the connected sum of two knots, cut each knot at any point and join the boundaries of the cut, keeping orientations consistent.

Let J and K be knots and Σ_J and Σ_K are Seifert surfaces

$$\chi(\Sigma_{K\#J}) = \chi(\Sigma_K) + \chi(\Sigma_J) - 2$$

$$g(J\#K) = g(J) + g(K)$$



Theorem: Let J and K be knots. Then,

$$g(J\#K) = g(J) + g(K)$$

Step 1:

$$g(J\#K) <= g(J) + g(K)$$

Let Σ_J and Σ_K be Seifert surfaces for knot J and K. Assume both have minimum genus so that,

$$g(\Sigma_J) = g(J) \qquad g(\Sigma_K) = g(K)$$

Euler characteristics of connected sum is,

$$\chi(\Sigma_{J\#K}) = \chi(\Sigma_J) + \chi(\Sigma_K) - 2$$

$$g(\Sigma_{j\#k}) = 1 - \chi(\Sigma_{j\#k})/2$$

$$= 1 - (\chi(\Sigma_j) + \chi(\Sigma_k) - 2)/2$$

$$= (2 - \chi(\Sigma_j))/2 + (2 - \chi(\Sigma_k))/2$$

$$= g(\Sigma_J) + g(\Sigma_k)$$

$$= g(J) + g(K)$$

Step 2: We are going to prove

$$g(K\#J) \geq g(K) + g(J)$$

Suppose Σ is a Seifert surface for K#J, such that

$$g(\Sigma) = g(K\#J)$$

By definition of connected sum, there is a 2-sphere S such that

$$S \cap \Sigma = \{A \cdot B\}$$

Now

$$S \cap \Sigma = \alpha \cup \beta_1 \cup \beta_2 .... \cup \beta_n$$

where α is an arc and

$$\partial\alpha = \{A, B\}$$

and β_1, β_2....β_n are pair-wise disjoint circles on S

We can arrange β_j's so that β_j bounds a disk $D_j \subseteq$ S and

$$D_j \cap \alpha = \phi, D_j \cap \beta_k = \phi$$

for

$$1 \leq k < j$$

Then we try to eliminate all β_j's
Since

$$D_n \cap \beta_k = \phi$$

for all k < j, we have

$$D_n \cap \Sigma = \beta_n = \partial D_n$$

Do surgery on Σ along the disk D_n

We get a surface $\Sigma_n^{'}$ out of Σ. Here are two possibilities:
1. $\Sigma_n^{'}$ is connected, then

$$g(\Sigma_n^{'}) = g(\Sigma_n) - 1$$

and

$$\partial\Sigma_n^{'} = K\#J$$

Contradiction.

2. Σ_n' has two components.

$$\Sigma_n' = \Sigma_n' \cup \Sigma_n''$$

Where, $\partial\Sigma_n = $ K#J

Σ_n'' has no boundary.

Then,

$$g(\Sigma_n') = g(\Sigma_n) + g(\Sigma_n'') \geq g(\Sigma_n)$$

So we must have

$$g(\Sigma_n) = g(\Sigma_n') = g(k\#J)$$

Yet

$$\Sigma_n \cap S = \alpha \cup \beta_1 \cup ..... \cup \beta_{n-1}$$

so we can keep doing surgeries and eliminate all β_j's:
The final case is,

$$\Sigma_1 \cap S = \alpha.$$

Cut Σ_1 along α. We get two surface Σ_K, Σ_j with

$$\partial\Sigma_k = K, \partial\Sigma_j = J$$

$$g(\Sigma) = g(\Sigma_1) = g(\Sigma_k) + g(\Sigma_J) >= g(K) + g(J)$$

Thank you to...

- Our parents, for driving us to MIT each week
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