On quasi-invariant polynomials

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Background and Motivation

Symmetric Polynomials

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- Examples:

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- Symmetric polynomials make a **ring**.

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$$\begin{aligned} P(x, y, z) = x^5 - x^4y + x^4z - 2x^3y^2 - 2x^3yz - 2x^2z^3 + xy^4 \\ - 2xy^3z - 2xyz^3 - xz^4 + y^5 - y^4z - 2y^3z^2 + yz^4 + z^5 \end{aligned}$$

$$P(x, y, z) - P(x, z, y) = (y - z)^3(y + z)(x + y + z)$$

Definitions

Quasi-invariant functions

Let m be a nonnegative integer. We say that a smooth function $F : \mathbb{C}^n \to \mathbb{C}$ is m-quasi-invariant if

$$\frac{F(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) - F(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)}{(x_i - x_j)^{2m+1}}$$

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Twisted quasi-invariants

Fix smooth functions $f_1, \ldots, f_n : \mathbb{C} \to \mathbb{C}$. We define $Q_m(f_1, \ldots, f_n)$ as the set of polynomials $F \in \mathbb{C}[x_1, \ldots, x_n]$ for which $f_1(x_1) \ldots f_n(x_n) F(x_1, \ldots, x_n)$ is m-quasi-invariant.

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Theorem (Braverman, Etingof, and Finkelberg)

If $f_i(x) = x^{a_i}$ where $a_i \in \mathbb{C}$ for all i and $a_i - a_j$ is not a nonzero integer for any i and j, then $Q_m(f_1, \dots, f_n)$ is free for all m.

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- Generalize to all smooth functions

Which $f_1, \ldots, f_n$ are interesting?

Theorem

If $dlog\left(\frac{f_i}{f_j}\right)$ *is not a rational function with complex coefficients, then* $(x_i - x_j)^{2m} \mid F$ *for any* $F \in Q_m$*. Conversely, any* $F \in (x_i - x_j)^{2m}\mathbb{C}[x_1, \ldots, x_n]$ *satisfies the condition for* i, j*.*

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- Focus on $n = 2$

Our main result ($n = 2$)

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Example

Every element in $Q_2(\sqrt{x-1}, 1)$ takes the form

$$r_1(x, y)(x^2 + 10xy + 5y^2 - 12x - 20y + 15)$$

$$+r_2(x, y)(x^3 + 21x^2y + 35xy^2 + 7y^3 - 24x^2 - 112xy - 56y^2 + 90x + 112y - 64)$$

where r_1 and r_2 are symmetric polynomials in x and y.

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Hilbert series

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$$H_m(t) = \sum_{d \geq 0} t^d \cdot \dim Q_{m,d}(f_1, \ldots, f_n)$$

where $Q_{m,d}(f_1, \ldots, f_n)$ is the subspace of Q_m consisting of polynomials with total degree d.

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- Hilbert series "measures the size" of Q_m

Hilbert series for $n = 2$

- Let $\mathrm{dlog}(f) = P(x) + \sum_{i=1}^{u} \sum_{j=1}^{v_i} \frac{b_{i,j}}{(x-a_i)^j}$

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Theorem

The Hilbert series of $Q_m(f, 1)$ is

$$\frac{t^{2m} + t^{2m+1} + \sum_{i=1}^{m} t^{2(m-i)+d_i(f)} - \sum_{i=1}^{m} t^{2(m-i)+d_i(f)+2}}{(1-t)(1-t^2)}$$

where $d_m(f) = m(\deg P + 1) + \sum_{i=1}^{u} d_{m,v_i}(a_i)$ and

$$d_{m,k}(z) := \begin{cases} \min(m, |z|) & \text{if } z \in \mathbb{Z} \text{ and } k = 1 \\ mk & \text{otherwise} \end{cases}.$$

Example

Let $f(x) = \frac{x^\pi \sqrt{x+1}}{(x+i)^2}$. The Hilbert series of $Q_3(f, 1)$ is

$$t^6 + 3t^7 + 6t^8 + 7t^9 + 8t^{10} + 9t^{11} + \dots$$

so the dimension of the $\mathbb{C}$ vector space of polynomials in Q_3 with degree 9 is 7.

Future prospects

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- Find the Hilbert series for $Q_m(f_1, \ldots, f_n)$
- Study q-deformations of Q_m

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