Simplicial Homology

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Brouwer's Fixed Point Theorem

Algebraic invariants have applications to topological problems.

Theorem (Brouwer)

Let D^n denote the closed unit ball in $\mathbb{R}^n$. Every continuous function from D^n to itself has a fixed point.

The proof uses the fact that retractable injections induce injections of homology groups: the existence of a fixed-point free endomorphism of D^n would imply that there is an injection

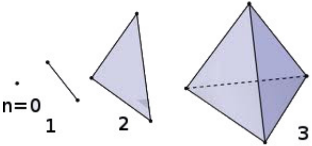
$$H_i(S^{n-1}, \mathbb{Z}) \hookrightarrow H_i(D^n, \mathbb{Z})$$

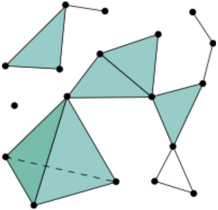
for all i, but

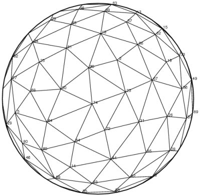
$$H_{n-1}(S^{n-1}, \mathbb{Z}) \cong \mathbb{Z} \text{ and } H_{n-1}(D^n, \mathbb{Z}) \cong 0.$$

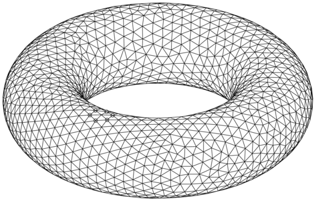
Triangulating spaces

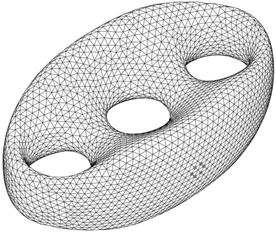
We think of an *n*-simplex as an *n*-dimensional triangle, and we can 'triangulate' a nice space by gluing a bunch of these together.











Boundary operators

We want to see 'holes' in our space. A hole is a place where "there could be something, but there isn't".

Write C_n for group of n-**chains**: integer linear combinations of n-simplices of a triangulated space. Define **boundary operators** $d_n : C_n \to C_{n-1}$ by

$$d_n s = \sum_{0 \leq i < n} (-1)^i s_i$$

for s an n-simplex. s_i is the ith face of s. Extended by linearity.

- A 1-simplex s is a line segment between two points, which are its 'faces'. If s goes from a to b, $d_1 s = b - a$.
- In particular, if $a = b$ (and s is really a loop) $d_1 s = 0$.

Cycles and boundaries

- If $d_n s = 0$, we say that $s \in C_n$ is a n-**cycle**. Cycles: 'could be something there'.
- If s is equal to $d_{n+1} t$ for some t, s is called an n-**boundary**. Boundaries: 'there is something there'. ("something" $= t$)
- Both the set B_n of n-boundaries and Z_n of n-cycles form subgroups of C_n, with $B_n \subset Z_n$.
- The nth **homology group** of the triangulated space is defined to be $H_n = Z_n / B_n$. Doesn't depend on triangulation.
- In some sense, this counts n-dimensional holes in the space: places where there could be an $(n+1)$-dimensional thing, but there isn't.

Chain complexes

Definition

A **chain complex** of vector spaces (modules, et cetera) is a sequence

$$\cdots \to_{d_{-2}} A_{-1} \to_{d_{-1}} A_0 \to_{d_0} A_1 \to_{d_1} \cdots$$

such that $d_{n+1} \circ d_n = 0$ for all n.

Definition

The *n***th cohomology group** of a chain complex $A_\bullet$ is

$$H^n(A) = \ker d_n / \operatorname{im} d_{n-1}.$$

Sheaves

Sheaves encode how locally defined functions glue together.

Definition

Let X be a topological space. A **sheaf $\mathcal{F}$ of sets on X** is the data of

1. for all open sets U, a set $\mathcal{F}(U)$;
2. for all open sets $U \subset V$, a function $res_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$,

such that

1. for all $U \subset V \subset W$, $res_{W,V} \circ res_{V,U} = res_{W,U}$
2. local sections glue together when they agree on the intersection of their domains of definition.

Examples of sheaves on $\mathbb{R}$

$\mathcal{F}(U) =$

- continuous real-valued functions on U
- smooth real-valued functions on U
- rational functions on U
- locally constant integer-valued functions on U (**constant sheaf** $\mathbb{Z}$)
- differential 1-forms on U

Sheaf cohomology

- Given a sheaf $\mathcal{F}$ of vector spaces (abelian groups) on a topological space X, one can cook up a chain complex, whose cohomology $H^i(X, \mathcal{F})$ defines the **sheaf cohomology groups of X with coefficients in $\mathcal{F}$.**
- These are the **derived functors** of the **global sections functor**, which associates to a sheaf $\mathcal{F}$ of abelian groups on X the abelian group $\mathcal{F}(X)$.
- This turns out to agree with simplicial cohomology – the simplicial cohomology groups with coefficents in an abelian group G are isomorphic to the sheaf cohomology groups with coefficients in $\underline{G}$. It is a useful topological invariant.

Application: the Exponential Exact Sequence

There is a diagram of sheaves on $\mathbb{C}$:

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow_f \mathcal{O} \longrightarrow_g \mathcal{O}^* \longrightarrow 0$$

where $\mathcal{O}(U)$ is holomorphic functions defined on U, $\mathcal{O}^*(U)$ is nonvanishing holomorphic functions on U, $f(n) = 2i\pi n$, and $g(f) = \exp(f)$. The image of each map is the kernel of the next. This gives a sequence (for U an open subset of $\mathbb{C}$)

$$\mathcal{O}(U) \longrightarrow \mathcal{O}^*(U) \longrightarrow H^1(U, \underline{\mathbb{Z}})$$

where again the image of the first morphism is the kernel of the second. Image: functions with global logarithms. $H^1(U, \underline{\mathbb{Z}})$ is simplicial cohomology— measures U's topology (holes).

Application: the Jordan Curve Theorem

Theorem (Jordan)

Let $f : S^{n-1} \hookrightarrow \mathbb{R}^n$ be an injective continuous function. Then, $\mathbb{R}^n \setminus \operatorname{im} f$ has two path-components.

The proof uses compactly supported cohomology (a variant of sheaf cohomology that is constructed by cooking up a different complex).

Acknowledgments

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Images:
http://brickisland.net/cs177fa12/
https://en.wikipedia.org/wiki/File:Tri-brezel.png
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