Removing Cycles from Dense Digraphs

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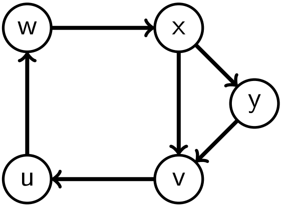
4th Annual MIT-PRIMES Conference

May 17, 2014

Directed graphs

- Digraph: directed graph, no multiple edges



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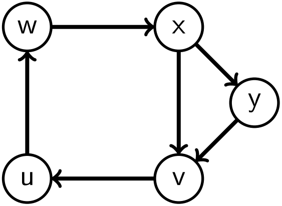

- Cycles in digraphs:

Definition

A digraph is *r-free* if the length of its shortest directed cycle is $> r$.

- Above graph is 3-free but not 4-free (i.e. *uwxv*).

Problems in r-free digraphs

Conjecture (Caccetta & Häggkvist, 1978)

Every r-free digraph on n vertices has a vertex of outdegree less than $\frac{n}{r}$.

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Parameters in r-free digraph G:

- $\beta(G)$: minimum number of edges needed to remove to make the graph acyclic (*minimum feedback arc set*).
- $\gamma(G)$: number of non-adjacent pairs of vertices.

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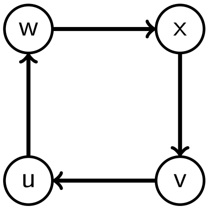
- Example: $\beta(C_4) = 1, \gamma(C_4) = 2$:



Bounds on $\beta(G)$

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If G is an r-free digraph, then $\beta(G) \leq \frac{2\gamma(G)}{(r-2)(r+1)}$.

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- $\beta(G) \leq 800\gamma(G)/r^2$.
- $\beta(G) \leq 25n^2/r^2$.

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Improved bounds:

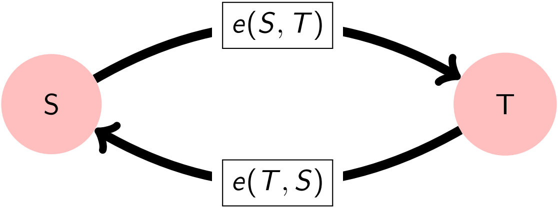
Theorem

If G is an r-free digraph on $n > 12$ vertices, then

- $\beta(G) < \frac{229\gamma(G)}{(r-2)^2}$.
- $\beta(G) < 5.59n^2/r^2$.

Edge expansion

Notation:



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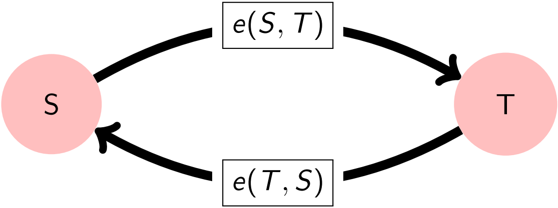

Definition

Edge expansion of a digraph G:

$$\mu(G) = \min_{\substack{S \subset V(G) \\ |S| \leq \frac{n}{2}}} \frac{\min\{e(S, V \setminus S), e(V \setminus S, S)\}}{|S|}.$$

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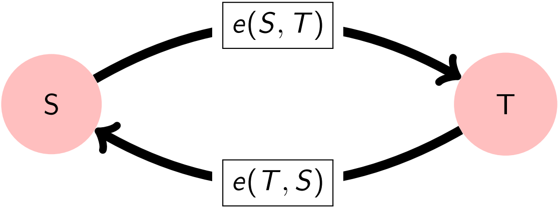

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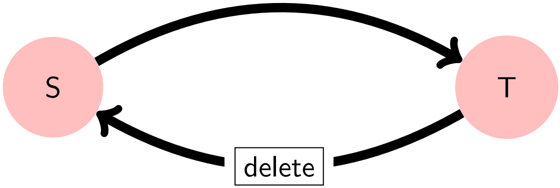
Modified edge expansion of a digraph G:

$$\lambda(G) = \min_{S \subset V(G)} \frac{\min\{e(S, V \backslash S), e(V \backslash S, S)\}}{|S| \cdot |V \backslash S|}.$$

Using results on edge expansion

- Big idea: choose S with $\lambda(S)$ small, let $T = V \backslash S$:



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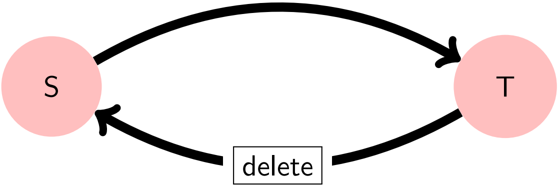

- Show that $\lambda(G) < 11.17/r^2 \Rightarrow \beta(G) < 5.59n^2/r^2$.

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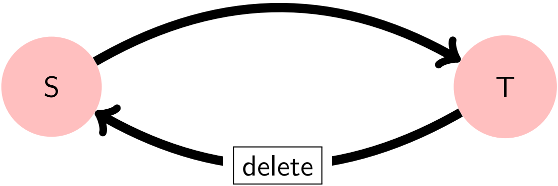

- Show that $\lambda(G) < 11.17/r^2 \Rightarrow \beta(G) < 5.59n^2/r^2$.
- Moreover, show that as $\gamma(G)$ decreases, $\lambda(G)$ decreases:
- If G is an r-free digraph, then $\beta(G) < \frac{229\gamma(G)}{(r-2)^2}$. *($\beta(G)$ is the size of the smallest feedback arc set.)*

The $r = 3$ case

Conjecture (Chudnovsky, Seymour & Sullivan, 2008)

If G is a 3-free digraph, then $\beta(G) \leq \gamma(G)/2$.

Previous work for $r = 3$:

$\beta(G) \leq \gamma(G)$ for 3-free digraphs	Chudnovsky, Seymour & Sullivan (2006)
$\beta(G) \leq .88\gamma(G)$ for 3-free digraphs	Dunkum, Hamburger & Pór (2009)
$\beta(G) \leq \gamma(G)/2$ for specific 3-free digraphs	Chudnovsky, Seymour & Sullivan (2006)

$\beta(G) \leq .88\gamma(G)$ used to improve bounds for Caccetta-Häggkvist conjecture.

Edge expansion of 3-free digraphs

Theorem

If G is a 3-free digraph, then $\lambda(G) \leq 1/3$.

- Idea: There is some subset of $V(G)$ with the number of edges coming out less than $1/3$ the total number of edges which could come out.
- Very weak version of Caccetta-Häggkvist conjecture: replace "vertex" with "subset of vertices".

Outline of proof

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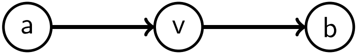

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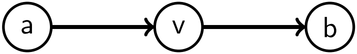
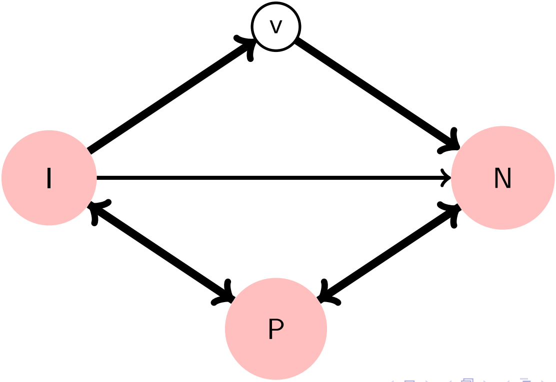


3. Consider λ of N = out-neighborhood of v, I = in-neighborhood of v:



Cayley graphs

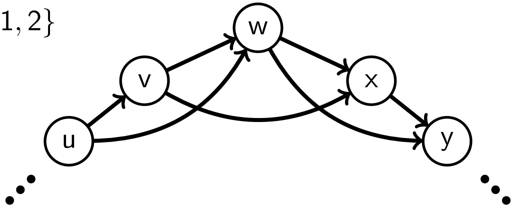
Definition

Given prime p, $A \subset \{1, \ldots, p-1\}$, form a *Cayley graph* on $v_0, \ldots, v_{p-1}$ by drawing edge between v_i, v_j if $j - i \pmod{p} \in A$.

Definition

Restricted Cayley graph: $A \subset \{1, \ldots, \frac{p-1}{2}\}$.

Example: $A = \{1, 2\}$



Caccetta-Häggkvist Conjecture proved for all Cayley graphs (Hamidoune, 1981).

Cayley graphs

- $\beta(G) \leq \gamma(G)/2$ open for all 3-free Cayley graphs on $\mathbb{Z}_p$.
- Known: if G is a Cayley graph on $\mathbb{Z}_p$, $\beta(G) \leq \gamma(G)/2$ if $|A| \leq (p-1)/4$. *($\beta(G)$ is size of minimum feedback arc set, $\gamma(G)$ is number of non-adjacent pairs of vertices.)*

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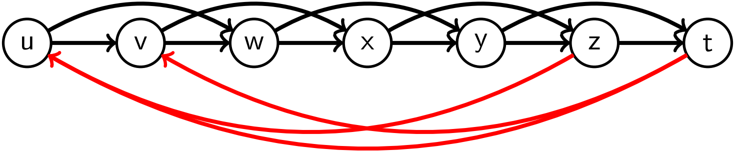
Theorem

If $p > 207$ is prime, and G is a 3-free restricted Cayley graph on $\mathbb{Z}_p$, then $\beta(G) \leq .4\gamma(G)$.

- $\beta(G) \leq \gamma(G)/3$ seems to be tight.
- Outline of proof:



Future work

- Improve bounds further.
- Consider more general Cayley graphs.
- Relate $\gamma(G)$, $\beta(G)$ to other parameters: e.g. if $\alpha(G) =$ number of distinct 4-cycles, then $\sqrt{\alpha(G)} \leq \frac{\gamma(G)}{2}$.

Acknowledgements

I would like to thank:

- László Miklós Lovász
- Professor Jacob Fox
- Dr. Tanya Khovanova
- MIT-PRIMES