Geodesics in the Hypercube

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Fourth Annual MIT-PRIMES Conference

May 17, 2014

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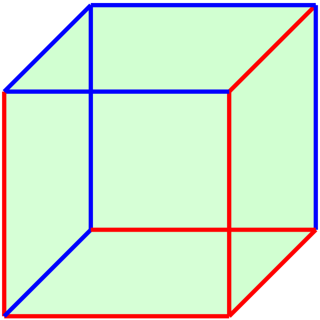

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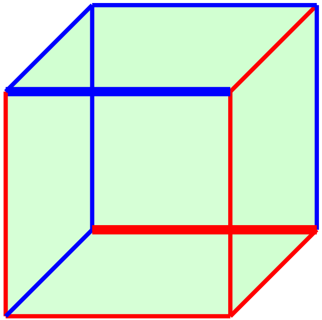

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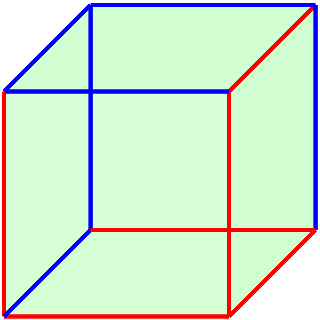

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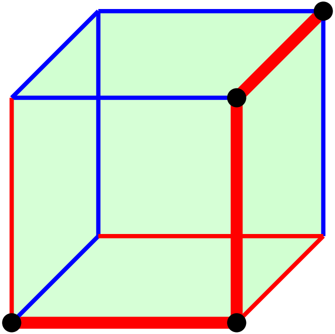

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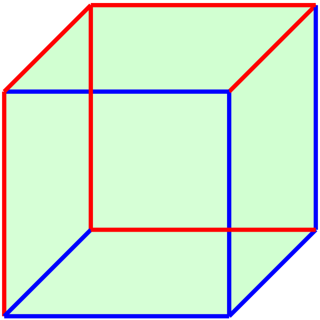

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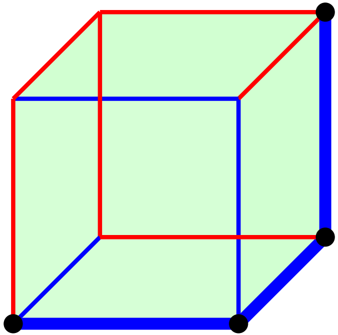

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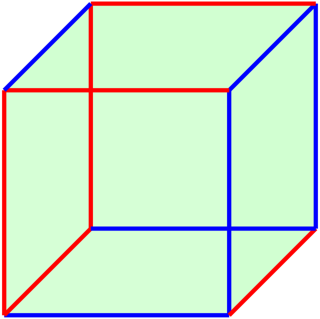

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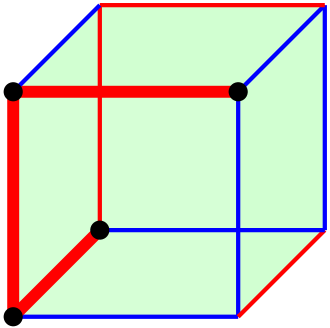

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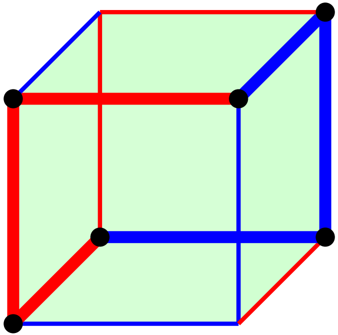

- Also notice how this monochromatic path cycles.

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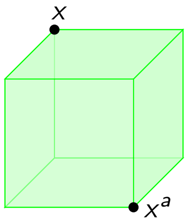

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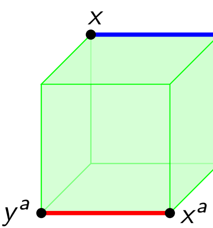

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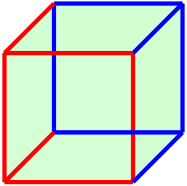

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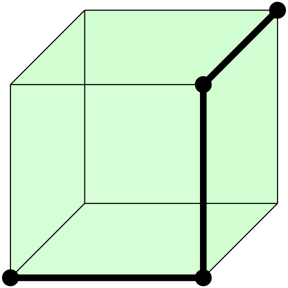

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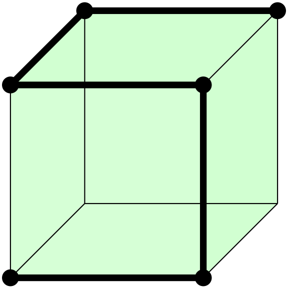

Conjectures

Conjecture (Leader and Long, 2013)

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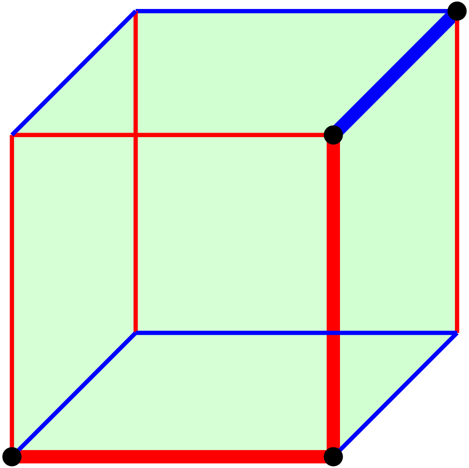
It has been shown that these two conjectures are equivalent.

Examples of Conjecture 2



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1. We showed that they were true for the cases $n = 2, 3, 4, 5, 6$.
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    1. Antipodal 2-colorings of the cube
    2. Subgraphs of the cube with a fixed proportion of edges

Maximality

1. Antipodal 2-colorings of Q_n
2. Subgraphs of Q_n with a fixed number of edges

Maximal Antipodal 2-colorings: Idea

We aim to *maximize* the number of monochromatic geodesics.

Definition

A **subcube** 2-**coloring** of Q_n colors the edges of disjoint $n-1$-dimensional subcubes in Q_n opposite colors, and then colors antipodally the remaining edges connecting these subcubes.

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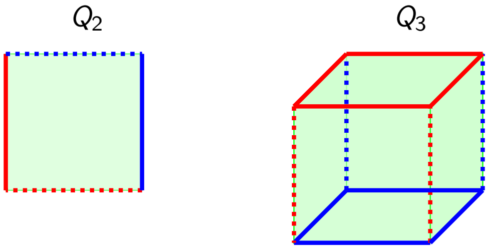

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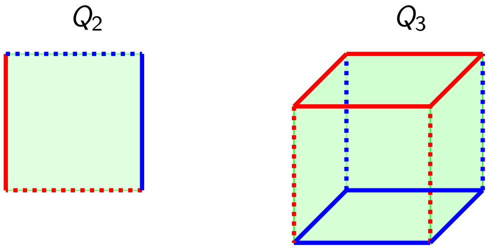

We conjectured that such a subcube coloring contained the maximum number of geodesics.

Antipodal 2-colorings: Optimality

Theorem

The maximum number of geodesics in an antipodal 2-coloring of Q_n is $2^{n-1}(n-1)!$, which occurs **only in a subcube coloring.**

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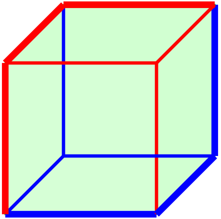

1. Antipodal 2-colorings of Q_n
2. Subgraphs of Q_n with a fixed number of edges

Subgraphs of the Cube: Idea

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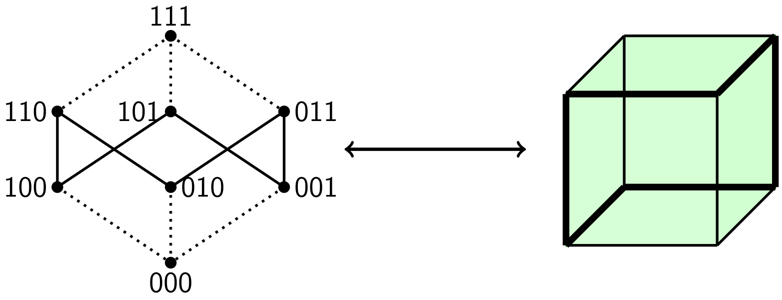
- Cycles have the most geodesics for the number of edges
- This led us to the configuration below: a subgraph containing all edges in the 'middle layer'



Subgraphs of the Cube

Let $d(v)$ be the number of 1's in the coordinate form of v.

Definition

A **middle-layer subgraph** is one containing an edge $E = \{v_1, v_2\} \in Q_n$ if and only if $\frac{n}{2} - C \leq d(v_1), d(v_2) \leq \frac{n}{2} + C$, where C depends on the proportion of edges.

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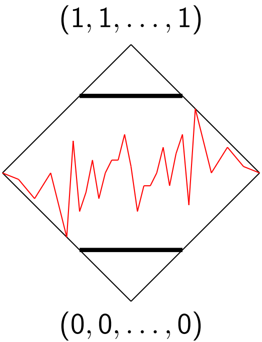
Edges concentrated in the 'middle layer'



Shown is a path of edges in this middle layer.

Subgraphs of the Cube: Computation

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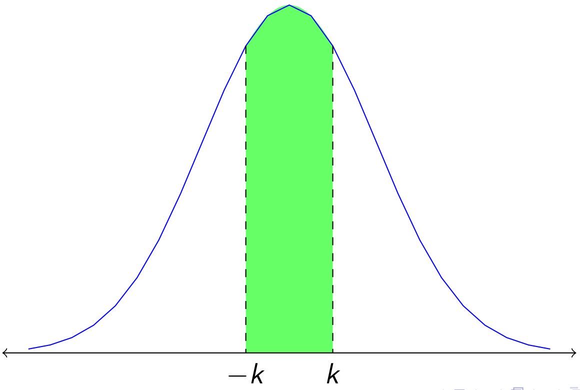
Result: Given that our proportion of edges is equivalent to the area shown below:



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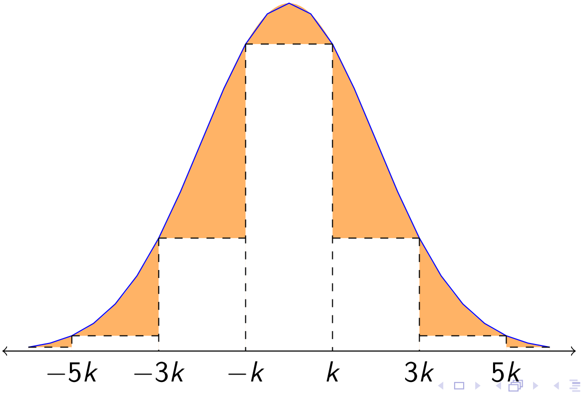

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- Incorporate probability into these colorings: e.g. the expected number of antipodal geodesics

Acknowledgements

Many thanks to:

- Yufei Zhao, my mentor
- MIT-PRIMES
- My awesome parents