Quotients of Lower Central Series With Multiple Relations

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Outline

What is a Non-commutative Algebra?

- Take 2 letters: x, y
- Words: xyx, $yyxyx$, $xyxyxyx$
- Sentences: $2xyx + 5yyxyx$
- This describes A_2
- Non-commutative letters: *logarithm* $\neq$ *algorithm*
- Why study Non-commutative algebras?
- Classical physics $\Longleftrightarrow$ Quantum physics
- Commutative algebras $\Longleftrightarrow$ Non-commutative algebras

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Free Algebra A_n

Definition

Free algebra:
$A_n := \mathbf{k}\langle x_1, x_2, x_3, \ldots, x_n \rangle$ and
$\{1, x_1, x_2, x_3, \ldots, x_1x_1, x_1x_2, x_1x_3, x_2x_1, \ldots\}$

- Free algebra is the most “non-commutative”
- A_n generated by generators $x_1, x_2, \ldots, x_n$
- Example: A_3:

$$1$$

$$x_1, x_2, x_3$$

$$x_1^2, x_1x_2, x_1x_3, x_2x_1, x_2^2, x_2x_3, x_3x_1, x_3x_2, x_3^2$$

- Grading keeps track of data better: $A_n[d]$
- $A_n[d]$ spanned by n^d basis elements

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Relations

- A_n already studied
- All finitely generated non-commutative algebras are quotients of A_n by relations
- Relations: e.g. $(xy = 0, \; y^2 = 0)$
- $A_2/(xy, y^2)$

$$\begin{gathered} 1 \\ x, y \\ x^2, yx \\ x^3, yx^2 \end{gathered}$$

- Restrict to homogeneous relations to preserve grading

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Commutator

Definition

Commutator: Let A be an algebra. The **commutator** of x and y is $[x, y] = xy - yx$. If H and K are subspaces of A, then $[H, K] = \text{span}\{[h, k] | h \in H \text{ and } k \in K\})$.

Lower Central Series

- Study A_n by Lower Central Series Filtration

Definition

Lower Central Series Filtration:
$A = L_1 \supseteq L_2 \supseteq L_3 \supseteq \dots$
Where $L_1 := A$ and $L_{k+1} := [L_k, A]$ for $k \geq 1$.

- L_{i+1} is the smallest subspace such that $a\ell = \ell a \mod L_{i+1}$ for all $\ell \in L_i$ and $a \in A$.
- Provides a measure of non-commutativity for the original algebra

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Ideal

- Study ideals M_k in order to preserve structures
- Sets and element: $Cd := \{cd | c \in C\}$

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Question

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1. How many basis elements are in each M_k in each degree?

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Quotient of the Ideals

- Shrinking is essentially "Difference"
- We take the quotient of these ideals:

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Quotient $N_k := M_k / M_{k+1}$

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Coefficients

- What coefficients do we work over for $A_2 = \mathbf{k}\langle x_1, x_2\rangle$?
- Option 1: Work over $\mathbf{k} = \mathbb{Z}$ make N_k Abelian groups ($\mathbb{Z}$ not field)
- Option 2: $\mathbf{k} = \mathbf{GF}(p)$ or $\mathbb{Q}$ make N_k vector spaces

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Abelian Groups

- Working with coefficients over $\mathbb{Z}$ yield Abelian groups
- Finitely generated Abelian groups decomposable:

$$\mathbb{A} = \mathbb{Z}^r \oplus \bigoplus_i \mathbb{Z}_i^{\alpha_i} \text{for prime powers } i.$$

- Data presented as $r(\prod(i^{\alpha_i}))$
- $\mathbb{Z}^3 \oplus \mathbb{Z}_2^4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5 \iff 3(2^4 \cdot 4 \cdot 5)$

Data on $N_k[d]$

- MAGMA calculations/data reconfiguration
- Previous work exists, modified to automate data collection
- Example: $\mathbb{Z}\langle x, y\rangle/(x^5, y^7)$

Table: $\mathbb{Z}\langle x, y\rangle/(x^2, y^5)$

$N_i[d]$	2	3	4	5	6	7	8	9	10
N_2	1	1(2)	1(2)	1(2)	$0(2\cdot5)$	0	0	0	0
N_3	0	2	3(2)	$3(2^2)$	$3(2^2)$	$1(2^2\cdot5^2)$	$0(2\cdot5)$	0	0
N_4	0	0	2	$3(2^2)$	$3(2^4)$	$2(2^6)$	$0(2^6)$	$0(2^3)$	0(2)
N_5	0	0	0	4	$7(2^3)$	$7(2^7)$	$5(2^{10})$	$1(2^{11})$	$0(2^7)$
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N_8	0	0	0	0	0	0	12	$24(2^{13})$	$20(2^{33}\cdot3\cdot4\cdot5\cdot7)$
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- Only 2 or 5 torsion appears except in $N_8[10]$

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Row Sums Over **GF**(p)

Proposition 1

Take $A_2/(x^m,\ y^n)$, where m is divisible by a prime p. If N_i is computed over **GF**(p), its total dimension is divisible by p. In fact, if m and n are both divisible by p, then the total dimension is divisible by p^2.

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Sums: 15, 45

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Another Example of Proposition 1

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- Sums: 9, 27

Proving Proposition 1

Theorem 1

If all relations are functions of x_1^p, then all $N_i(A)$ carry an action of the Weyl algebra $D(\mathbf{k})$ with generators D,x and relations $[D, x] = 1$.

- Proof sketch: Define D acting on an element a as $D(a) = \frac{d}{dx_1}a$ and $xa = x_1 a$. We can verify that $[D, x] = 1$.

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- Proof sketch: Define D acting on an element a as $D(a) = \frac{d}{dx_1}a$ and $xa = x_1 a$. We can verify that $[D, x] = 1$.

Basic Corollary of Theorem 1

Corollary 2

If $N_i(A)$ is finite dimensional, then $\dim(N_i(A))$ is divisible by p. In general, if the relations are non-commutative polynomials of p-th powers of the first r variables $x_1^p, x_2^p, \ldots, x_r^p$, then $\dim(N_i(A))$ is divisible by p^r.

- Proof sketch: We use $k = \mathbf{GF}(p)$. $0 = Tr([D,x]) = Tr(1) = dim(V)$ in $\mathbf{GF}(p)$ where V is a representation of $D(k)$, so every representation of $D(k)$ has dimension divisible by p. For relations which are functions of $x_1^p, x_2^p, \ldots, x_r^p$, we have an action of the tensor power algebra $D(\mathbf{k})^{\otimes r}$ whose representation dimensions are divisible by p^r.

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Further Corollary of Theorem 1

Corollary 3

Suppose that the relations are homogeneous in x_i. Then, the Hilbert series H of $N_i(A)$ with respect to $X_1, \ldots, X_r$ is divisible by $P_r := (1 + X_1 + \ldots + X_1^{p-1}) \ldots (1 + X_r + \ldots + X_r^{p-1})$, i.e. $\frac{H}{P_r}$ is a power series with non-negative coefficients.

Further Research Options

- Bigrading: $N_k[d_x, d_y]$ instead of $N_k[d]$
- Example: xy^2x^3 has bidegree $(4, 2)$, $yxyx$ has bidegree $(2, 2)$
- Instead of N_k, find B_k (Quotients of L_k/L_{k+1})
- Use A_3 or even A_4

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