Beyond Alternating Permutations: Pattern Avoidance in Young Diagrams and Tableaux

Nihal Gowravaram and Ravi Jagadeesan
MIT PRIMES

Mentor: Joel Lewis

May 19, 2012

Alternating Permutations

- We will treat a permutation $w \in S_n$ as a sequence $w_1, w_2, \cdots, w_n$ containing every positive integer $k \leq n$ exactly once.

Alternating Permutations

- We will treat a permutation $w \in S_n$ as a sequence $w_1, w_2, \cdots, w_n$ containing every positive integer $k \leq n$ exactly once.
- A permutation w is called *alternating* if

$$w_1 < w_2 > w_3 < w_4 > \cdots .$$

Alternating Permutations

- We will treat a permutation $w \in S_n$ as a sequence $w_1, w_2, \cdots, w_n$ containing every positive integer $k \leq n$ exactly once.
- A permutation w is called *alternating* if

$$w_1 < w_2 > w_3 < w_4 > \cdots.$$

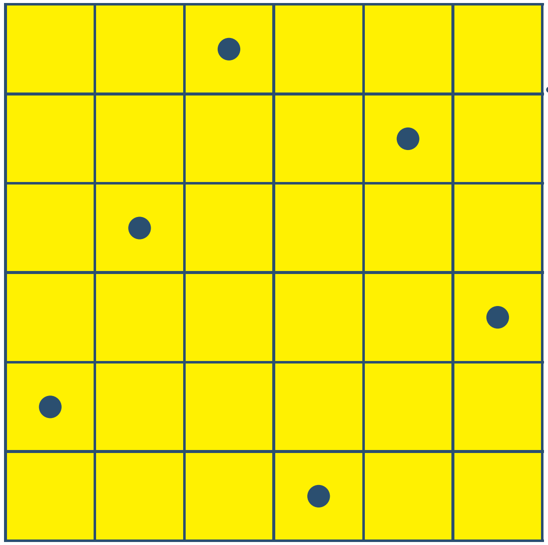
For example, 352614 is alternating.

Alternating Permutations

- We will treat a permutation $w \in S_n$ as a sequence $w_1, w_2, \cdots, w_n$ containing every positive integer $k \leq n$ exactly once.
- A permutation w is called *alternating* if

$$w_1 < w_2 > w_3 < w_4 > \cdots .$$

For example, 352614 is alternating. Graphically, this is



.

- A permutation w is said to *contain* a permutation q if there is a subsequence of w order-isomorphic to q. If w does not contain q, then w *avoids* q.

Pattern Containment in Permutations

- A permutation w is said to *contain* a permutation q if there is a subsequence of w order-isomorphic to q. If w does not contain q, then w *avoids* q.
  For example, 3**25**64**1** contains 231.

Pattern Containment in Permutations

- A permutation w is said to *contain* a permutation q if there is a subsequence of w order-isomorphic to q. If w does not contain q, then w *avoids* q.
  For example, 3**25**6**41** contains 231.
- Given a permutation q and a positive integer n, let $S_n(q)$ ($A_n(q)$) denote the set of all (alternating) permutations of length n that avoid q.

Pattern Containment in Permutations

- A permutation w is said to *contain* a permutation q if there is a subsequence of w order-isomorphic to q. If w does not contain q, then w *avoids* q.
  For example, 3**25**6**41** contains 231.
- Given a permutation q and a positive integer n, let $S_n(q)$ ($A_n(q)$) denote the set of all (alternating) permutations of length n that avoid q.
- If $|S_n(p)| = |S_n(q)|$ for all n, we say that p and q are *Wilf-equivalent*.

Pattern Containment in Permutations

- A permutation w is said to *contain* a permutation q if there is a subsequence of w order-isomorphic to q. If w does not contain q, then w *avoids* q.
  For example, 3**25**6**41** contains 231.
- Given a permutation q and a positive integer n, let $S_n(q)$ ($A_n(q)$) denote the set of all (alternating) permutations of length n that avoid q.
- If $|S_n(p)| = |S_n(q)|$ for all n, we say that p and q are *Wilf-equivalent*.
- If $|A_n(p)| = |A_n(q)|$ for all n, we say that p and q are *equivalent for alternating permutations*.

Previous Results

- (Mansour, Deutsch, Reifegerste) If q is a pattern of length 3, then $|A_n(q)|$ is a Catalan number (i.e. of the form $C_k = \frac{(2k)!}{k!(k+1)!}$).
  The indices depend on the choice of q and on the parity of n.

- (Mansour, Deutsch, Reifegerste) If q is a pattern of length 3, then $|A_n(q)|$ is a Catalan number (i.e. of the form $C_k = \frac{(2k)!}{k!(k+1)!}$).
- (Lewis) For patterns of length 4,

$$|A_{2n}(1234)| = |A_{2n}(2143)| = \frac{2(3n)!}{n!(n+1)!(n+2)!},$$

$$|A_{2n+1}(1234)| = \frac{16(3n)!}{(n-1)!(n+1)!(n+3)!},$$

$$|A_{2n+1}(2143)| = \frac{2(3n+3)!}{n!(n+1)!(n+2)!(2n+1)(2n+2)(2n+3)}.$$

The Main Theorem and Its Motivation

Theorem (Backelin-West-Xin). *For all $t \geq k$ and all permutations q of $\{k+1, k+2, k+3, \cdots, t\}$, the patterns $123 \cdots kq$ and $k(k-1)(k-2) \cdots 1q$ are Wilf-Equivalent.*

Theorem (Backelin-West-Xin). *For all $t \geq k$ and all permutations q of $\{k+1, k+2, k+3, \cdots, t\}$, the patterns $123 \cdots kq$ and $k(k-1)(k-2) \cdots 1q$ are Wilf-Equivalent.*

Main Results:

For all q, the following sets of patterns are equivalent for alternating permutations.

- $12q$ and $21q$

Theorem (Backelin-West-Xin). *For all $t \geq k$ and all permutations q of $\{k+1, k+2, k+3, \cdots, t\}$, the patterns $123\cdots kq$ and $k(k-1)(k-2)\cdots 1q$ are Wilf-Equivalent.*

Main Results:

For all q, the following sets of patterns are equivalent for alternating permutations.

- $12q$ and $21q$
- $123q$, $213q$ and $321q$

Theorem (Backelin-West-Xin). *For all $t \geq k$ and all permutations q of $\{k+1, k+2, k+3, \cdots, t\}$, the patterns $123\cdots kq$ and $k(k-1)(k-2)\cdots 1q$ are Wilf-Equivalent.*

Main Results:

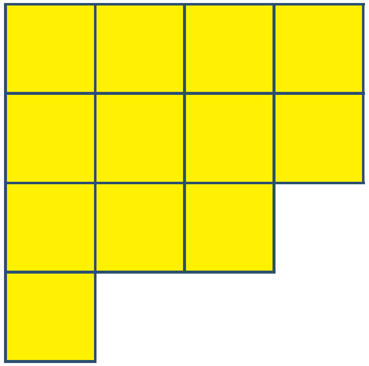
For all q, the following sets of patterns are equivalent for alternating permutations.

- $12q$ and $21q$
- $123q$, $213q$ and $321q$
- (Conjecture) For all k, $123\cdots kq$ and $k(k-1)(k-2)\cdots 1q$

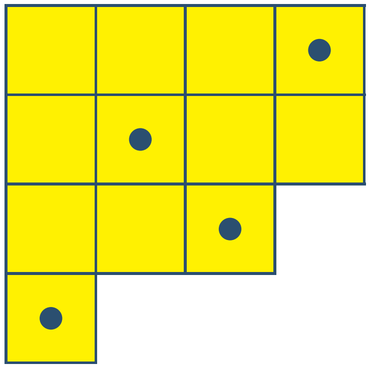
Pattern Avoidance of Young Diagrams

Basic Definitions

- A *Young diagram* with n rows/columns is a set Y of squares of an $n \times n$ board such that if a square $S \in Y$, then any square above and to the left of S is also in Y.

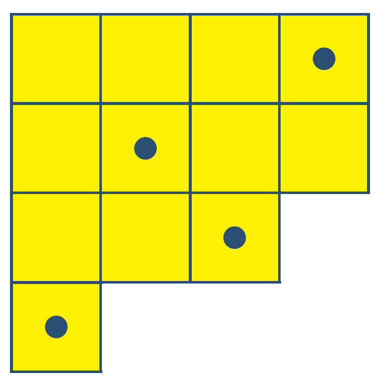


- A *Young diagram* with n rows/columns is a set Y of squares of an $n \times n$ board such that if a square $S \in Y$, then any square above and to the left of S is also in Y.
- A *transversal* T of Y is a set of squares of Y that contains exactly one member per row and per column of Y.



Basic Definitions

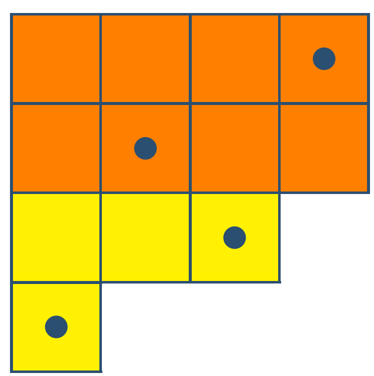
- A *Young diagram* with n rows/columns is a set Y of squares of an $n \times n$ board such that if a square $S \in Y$, then any square above and to the left of S is also in Y.
- A *transversal* T of Y is a set of squares of Y that contains exactly one member per row and per column of Y.
- T is said to *contain* a $k \times k$ permutation matrix $M = (m_{i,j})$ if there are k rows $r_1 < r_2 < \cdots < r_k$ and k columns $c_1 < c_2 < \cdots < c_k$ of Y such that $(r_k, c_k) \in Y$ and $(r_i, c_j) \in T$ if and only if the entry of $m_{i,j} = 1$.



contains $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Basic Definitions

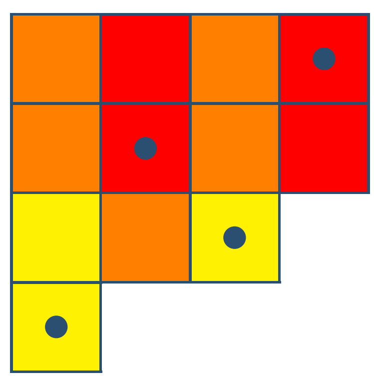
- A *Young diagram* with n rows/columns is a set Y of squares of an $n \times n$ board such that if a square $S \in Y$, then any square above and to the left of S is also in Y.
- A *transversal* T of Y is a set of squares of Y that contains exactly one member per row and per column of Y.
- T is said to *contain* a $k \times k$ permutation matrix $M = (m_{i,j})$ if there are k rows $r_1 < r_2 < \cdots < r_k$ and k columns $c_1 < c_2 < \cdots < c_k$ of Y such that $(r_k, c_k) \in Y$ and $(r_i, c_j) \in T$ if and only if the entry of $m_{i,j} = 1$.



contains $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Basic Definitions

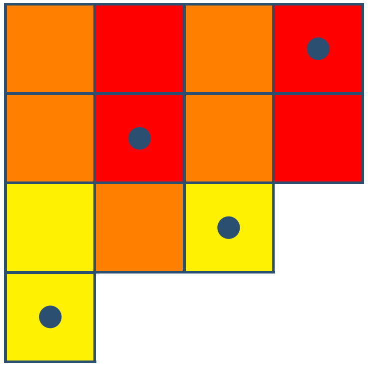
- A *Young diagram* with n rows/columns is a set Y of squares of an $n \times n$ board such that if a square $S \in Y$, then any square above and to the left of S is also in Y.
- A *transversal* T of Y is a set of squares of Y that contains exactly one member per row and per column of Y.
- T is said to *contain* a $k \times k$ permutation matrix $M = (m_{i,j})$ if there are k rows $r_1 < r_2 < \cdots < r_k$ and k columns $c_1 < c_2 < \cdots < c_k$ of Y such that $(r_k, c_k) \in Y$ and $(r_i, c_j) \in T$ if and only if the entry of $m_{i,j} = 1$.



contains $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Basic Definitions

- A *Young diagram* with n rows/columns is a set Y of squares of an $n \times n$ board such that if a square $S \in Y$, then any square above and to the left of S is also in Y.
- A *transversal* T of Y is a set of squares of Y that contains exactly one member per row and per column of Y.
- T is said to *contain* a $k \times k$ permutation matrix $M = (m_{i,j})$ if there are k rows $r_1 < r_2 < \cdots < r_k$ and k columns $c_1 < c_2 < \cdots < c_k$ of Y such that $(r_k, c_k) \in Y$ and $(r_i, c_j) \in T$ if and only if the entry of $m_{i,j} = 1$.

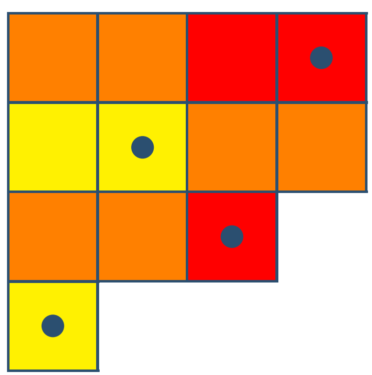


contains $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

All 4 red squares are in the Young diagram.

Basic Definitions

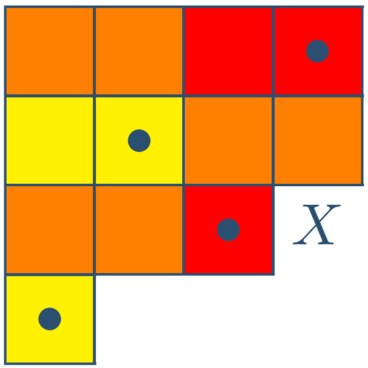
- A *Young diagram* with n rows/columns is a set Y of squares of an $n \times n$ board such that if a square $S \in Y$, then any square above and to the left of S is also in Y.
- A *transversal* T of Y is a set of squares of Y that contains exactly one member per row and per column of Y.
- T is said to *contain* a $k \times k$ permutation matrix $M = (m_{i,j})$ if there are k rows $r_1 < r_2 < \cdots < r_k$ and k columns $c_1 < c_2 < \cdots < c_k$ of Y such that $(r_k, c_k) \in Y$ and $(r_i, c_j) \in T$ if and only if the entry of $m_{i,j} = 1$.



is not a copy of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Basic Definitions

- A *Young diagram* with n rows/columns is a set Y of squares of an $n \times n$ board such that if a square $S \in Y$, then any square above and to the left of S is also in Y.
- A *transversal* T of Y is a set of squares of Y that contains exactly one member per row and per column of Y.
- T is said to *contain* a $k \times k$ permutation matrix $M = (m_{i,j})$ if there are k rows $r_1 < r_2 < \cdots < r_k$ and k columns $c_1 < c_2 < \cdots < c_k$ of Y such that $(r_k, c_k) \in Y$ and $(r_i, c_j) \in T$ if and only if the entry of $m_{i,j} = 1$.



is not a copy of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

The square X is not in the Young diagram.

Basic Definitions

- A *Young diagram* with n rows/columns is a set Y of squares of an $n \times n$ board such that if a square $S \in Y$, then any square above and to the left of S is also in Y.
- A *transversal* T of Y is a set of squares of Y that contains exactly one member per row and per column of Y.
- T is said to *contain* a $k \times k$ permutation matrix $M = (m_{i,j})$ if there are k rows $r_1 < r_2 < \cdots < r_k$ and k columns $c_1 < c_2 < \cdots < c_k$ of Y such that $(r_k, c_k) \in Y$ and $(r_i, c_j) \in T$ if and only if the entry of $m_{i,j} = 1$. Otherwise, we say that T *avoids* M.

Basic Definitions

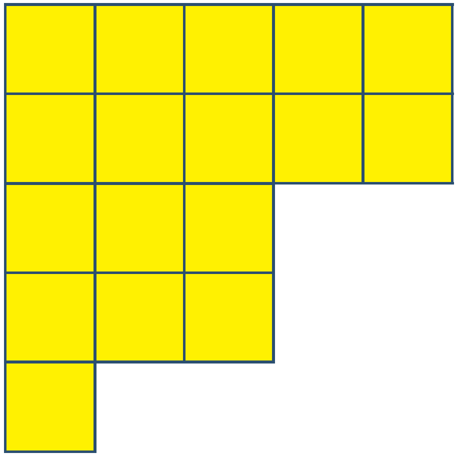
- A *Young diagram* with n rows/columns is a set Y of squares of an $n \times n$ board such that if a square $S \in Y$, then any square above and to the left of S is also in Y.
- A *transversal* T of Y is a set of squares of Y that contains exactly one member per row and per column of Y.
- T is said to *contain* a $k \times k$ permutation matrix $M = (m_{i,j})$ if there are k rows $r_1 < r_2 < \cdots < r_k$ and k columns $c_1 < c_2 < \cdots < c_k$ of Y such that $(r_k, c_k) \in Y$ and $(r_i, c_j) \in T$ if and only if the entry of $m_{i,j} = 1$. Otherwise, we say that T *avoids* M.
- If permutation matrices M and M' are such that, for all Young diagrams Y, the number of transversals of Y avoiding M is the same as the number avoiding M', we say that M and M' are *shape-Wilf equivalent*.

Ascents and Descents in Young Diagrams

- Given a transversal $T = \{(i, b_i)\}$ of a Young diagram, we say that i is *an ascent of* T (descent) when it is an ascent (descent) of $b_1 b_2 \cdots$.

Ascents and Descents in Young Diagrams

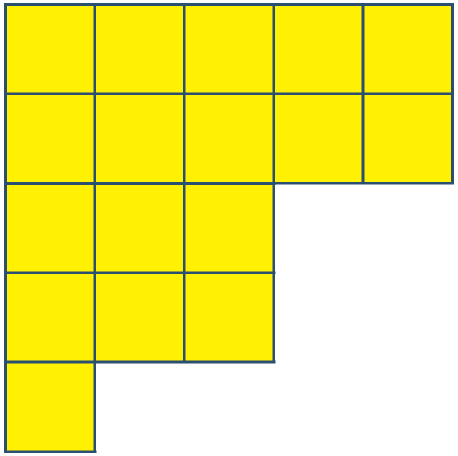
- Given a transversal $T = \{(i, b_i)\}$ of a Young diagram, we say that i is *an ascent of T* (descent) when it is an ascent (descent) of $b_1 b_2 \cdots$.
- An *AD-Young diagram* is a triple $\mathcal{Y} = (Y, A, D)$ of a Young diagram Y with n rows, and disjoint sets $A, D \subseteq [n-1]$ such that if $i \in A \cup D$, then the ith and $(i+1)$st rows of Y have the same length.

$Y =$ 

Ascents and Descents in Young Diagrams

- Given a transversal $T = \{(i, b_i)\}$ of a Young diagram, we say that i is *an ascent of* T (descent) when it is an ascent (descent) of $b_1 b_2 \cdots$.
- An *AD-Young diagram* is a triple $\mathcal{Y} = (Y, A, D)$ of a Young diagram Y with n rows, and disjoint sets $A, D \subseteq [n-1]$ such that if $i \in A \cup D$, then the ith and $(i+1)$st rows of Y have the same length.

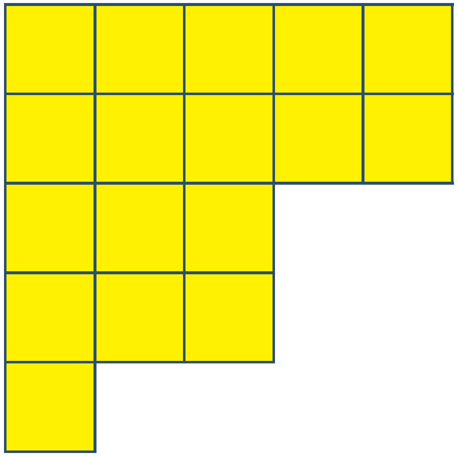
$Y =$



$A = \{1\}$ $\qquad$ $D = \{3\}$

Ascents and Descents in Young Diagrams

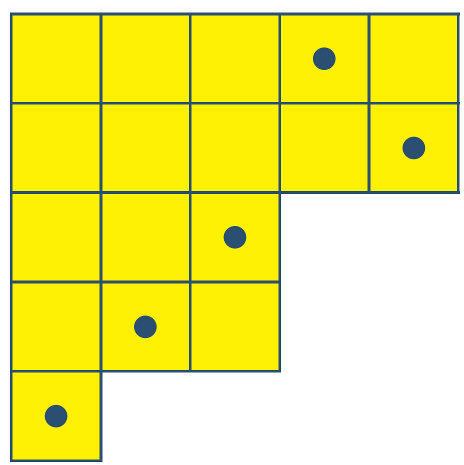
- Given a transversal $T = \{(i, b_i)\}$ of a Young diagram, we say that i is *an ascent of* T (descent) when it is an ascent (descent) of $b_1 b_2 \cdots$.
- An *AD-Young diagram* is a triple $\mathcal{Y} = (Y, A, D)$ of a Young diagram Y with n rows, and disjoint sets $A, D \subseteq [n-1]$ such that if $i \in A \cup D$, then the ith and $(i+1)$st rows of Y have the same length.
- A *valid transversal* of $\mathcal{Y}$ is a transversal T of Y such that if $i \in A$ (D), then i is an ascent (descent) of T.

$Y =$ 

$A = \{1\}$ $D = \{3\}$

Ascents and Descents in Young Diagrams

- Given a transversal $T = \{(i, b_i)\}$ of a Young diagram, we say that i is *an ascent of* T (descent) when it is an ascent (descent) of $b_1 b_2 \cdots$.
- An *AD-Young diagram* is a triple $\mathcal{Y} = (Y, A, D)$ of a Young diagram Y with n rows, and disjoint sets $A, D \subseteq [n-1]$ such that if $i \in A \cup D$, then the ith and $(i+1)$st rows of Y have the same length.
- A *valid transversal* of $\mathcal{Y}$ is a transversal T of Y such that if $i \in A$ (D), then i is an ascent (descent) of T.

$Y =$  $A = \{1\}$ $D = \{3\}$

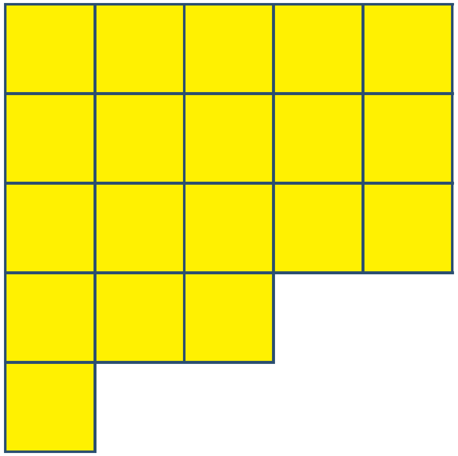
Ascents and Descents in Young Diagrams

- Given a transversal $T = \{(i, b_i)\}$ of a Young diagram, we say that i is *an ascent of* T (descent) when it is an ascent (descent) of $b_1 b_2 \cdots$.
- An *AD-Young diagram* is a triple $\mathcal{Y} = (Y, A, D)$ of a Young diagram Y with n rows, and disjoint sets $A, D \subseteq [n-1]$ such that if $i \in A \cup D$, then the ith and $(i+1)$st rows of Y have the same length.
- A *valid transversal* of $\mathcal{Y}$ is a transversal T of Y such that if $i \in A$ (D), then i is an ascent (descent) of T. Pattern avoidance is exactly as in Young diagrams.

- Given a transversal $T = \{(i, b_i)\}$ of a Young diagram, we say that i is *an ascent of* T (descent) when it is an ascent (descent) of $b_1 b_2 \cdots$.
- An *AD-Young diagram* is a triple $\mathcal{Y} = (Y, A, D)$ of a Young diagram Y with n rows, and disjoint sets $A, D \subseteq [n-1]$ such that if $i \in A \cup D$, then the ith and $(i+1)$st rows of Y have the same length.
- A *valid transversal* of $\mathcal{Y}$ is a transversal T of Y such that if $i \in A$ (D), then i is an ascent (descent) of T. Pattern avoidance is exactly as in Young diagrams.
- Given a permutation matrix M and an AD-Young diagram $\mathcal{Y}$, let $S_{\mathcal{Y}}(M)$ denote the set of valid transversals of $\mathcal{Y}$ that avoid M.

Alternating AD-Young Diagrams

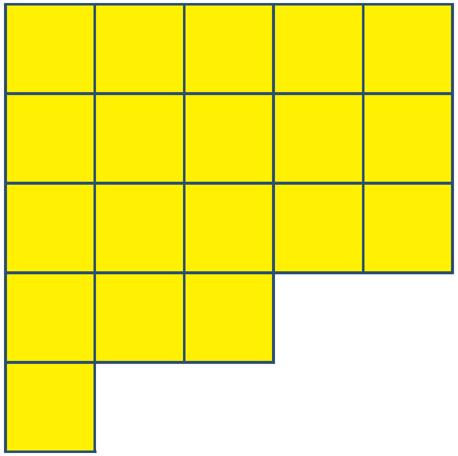
- An AD-Young diagram $\mathcal{Y} = (Y, A, D)$ with Y a Young diagram with n columns is called x-*alternating* if it satisfies the property that if $i \leq n - x$, then $i \in A$ if and only if $i + 1 \in D$.

$Y =$ 

Alternating AD-Young Diagrams

- An AD-Young diagram $\mathcal{Y} = (Y, A, D)$ with Y a Young diagram with n columns is called x-*alternating* if it satisfies the property that if $i \leq n - x$, then $i \in A$ if and only if $i + 1 \in D$.

$Y =$

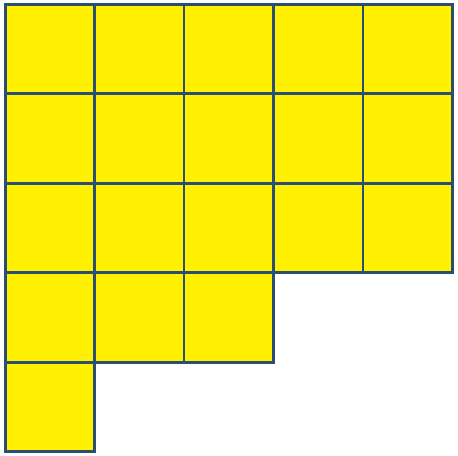


$A = \{1\}$

$D = \{2\}$

Alternating AD-Young Diagrams

- An AD-Young diagram $\mathcal{Y} = (Y, A, D)$ with Y a Young diagram with n columns is called x-*alternating* if it satisfies the property that if $i \leq n - x$, then $i \in A$ if and only if $i + 1 \in D$.

$Y =$ 

$A = \{1\}$

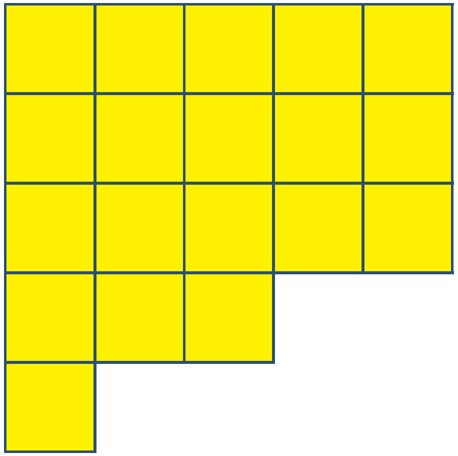
$D = \{2\}$

is 1-alternating.

Alternating AD-Young Diagrams

- An AD-Young diagram $\mathcal{Y} = (Y, A, D)$ with Y a Young diagram with n columns is called x-*alternating* if it satisfies the property that if $i \leq n - x$, then $i \in A$ if and only if $i + 1 \in D$.

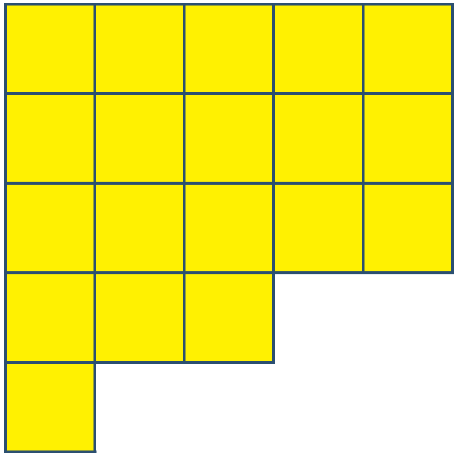
$Y =$



Alternating AD-Young Diagrams

- An AD-Young diagram $\mathcal{Y} = (Y, A, D)$ with Y a Young diagram with n columns is called x-*alternating* if it satisfies the property that if $i \leq n - x$, then $i \in A$ if and only if $i + 1 \in D$.

$Y =$

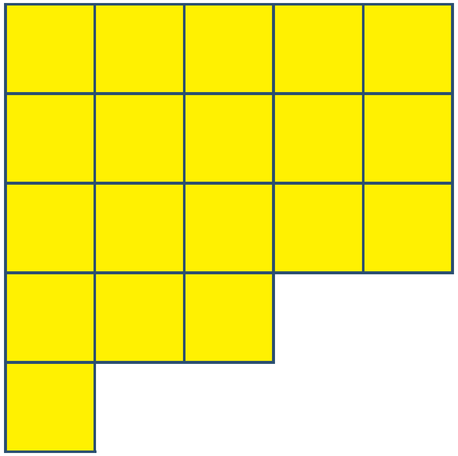


$A = \{2\}$

$D = \emptyset$

- An AD-Young diagram $\mathcal{Y} = (Y, A, D)$ with Y a Young diagram with n columns is called x-*alternating* if it satisfies the property that if $i \leq n - x$, then $i \in A$ if and only if $i + 1 \in D$.

$Y =$



$A = \{2\}$

$D = \emptyset$

is 4-alternating.

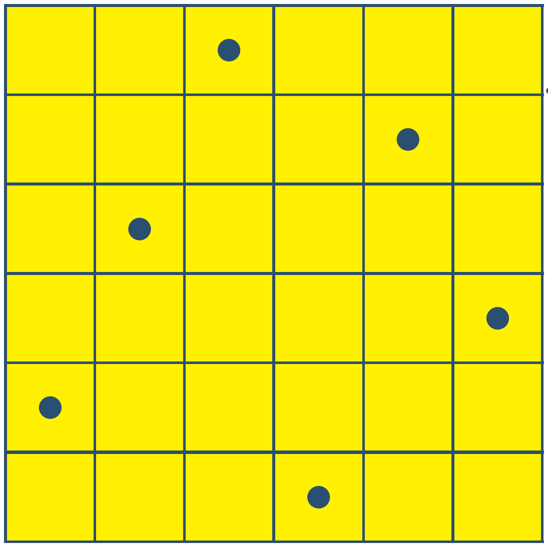
- An AD-Young diagram $\mathcal{Y} = (Y, A, D)$ with Y a Young diagram with n columns is called x-*alternating* if it satisfies the property that if $i \le n - x$, then $i \in A$ if and only if $i + 1 \in D$.
- If M and M' are permutation matrices such that for all x-alternating AD-Young diagrams $\mathcal{Y}$, we have $|S_{\mathcal{Y}}(M)| = |S_{\mathcal{Y}}(M')|$, then we say that M and M' are *shape-equivalent for* x*-alternating AD-Young diagrams.*

Alternating Permutations as Transversals

- We can treat a permutation b of length n as a transversal $\{(i, b_i)\}$ of the $n \times n$ Young diagram.

Alternating Permutations as Transversals

- We can treat a permutation b of length n as a transversal $\{(i, b_i)\}$ of the $n \times n$ Young diagram.
- We can treat an alternating permutation b of length $2n$ as a valid transversal $\{(i, b_i)\}$ of the 2-alternating AD-Young diagram (Y, A, D) with Y the $2n \times 2n$ square, $A = \{1, 3, 5, \cdots, 2n-1\}$, and $D = \{2, 4, 6, \cdots, 2n-2\}$. The permutation 352614 is



.

- We can treat a permutation b of length n as a transversal $\{(i, b_i)\}$ of the $n \times n$ Young diagram.
- We can treat an alternating permutation b of length $2n$ as a valid transversal $\{(i, b_i)\}$ of the 2-alternating AD-Young diagram (Y, A, D) with Y the $2n \times 2n$ square, $A = \{1, 3, 5, \cdots, 2n-1\}$, and $D = \{2, 4, 6, \cdots, 2n-2\}$.
- A permutation b avoids a pattern q if and only if its corresponding transversal avoids q's permutation matrix.

Alternating Permutations as Transversals

- We can treat a permutation b of length n as a transversal $\{(i, b_i)\}$ of the $n \times n$ Young diagram.
- We can treat an alternating permutation b of length $2n$ as a valid transversal $\{(i, b_i)\}$ of the 2-alternating AD-Young diagram (Y, A, D) with Y the $2n \times 2n$ square, $A = \{1, 3, 5, \cdots, 2n-1\}$, and $D = \{2, 4, 6, \cdots, 2n-2\}$.
- A permutation b avoids a pattern q if and only if its corresponding transversal avoids q's permutation matrix.
- Similarly, alternating permutations of odd length, can be treated as valid transversals of 1-alternating AD-Young diagrams.

Extending Alternating Shape-Equivalences

Theorem (Babson-West). *If M and M' are permutation matrices that are shape-Wilf equivalent, and P is an permutation matrix of positive dimensions, then the matrices*

$$\begin{bmatrix} M & 0 \\ 0 & P \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} M' & 0 \\ 0 & P \end{bmatrix}$$

are shape-Wilf equivalent.

Theorem (Babson-West). *If M and M' are permutation matrices that are shape-Wilf equivalent, and P is an permutation matrix of positive dimensions, then the matrices*

$$\begin{bmatrix} M & 0 \\ 0 & P \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} M' & 0 \\ 0 & P \end{bmatrix}$$

are shape-Wilf equivalent.

Theorem. *If M and M' are permutation matrices that are shape-Equivalent for x-alternating AD-Young diagrams, and P is an $r \times r$ permutation matrix, then the matrices*

$$\begin{bmatrix} M & 0 \\ 0 & P \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} M' & 0 \\ 0 & P \end{bmatrix}$$

are shape-equivalent for $x + r$-alternating AD-Young diagrams.

Theorem (Backelin-West-Xin). *For all k, the permutation matrices of the permutations $(k-1)(k-2)(k-3)\cdots 1k$ and $k(k-1)(k-2)\cdots 1$ are shape-Wilf equivalent.*

The Main Theorem Revisited

Theorem (Backelin-West-Xin). *For all k, the permutation matrices of the permutations $(k-1)(k-2)(k-3)\cdots 1k$ and $k(k-1)(k-2)\cdots 1$ are shape-Wilf equivalent.*

Theorem. *The permutation matrices corresponding to the permutations 12 and 21 are shape-equivalent for 1-alternating AD-Young diagrams.*

Theorem (Backelin-West-Xin). *For all k, the permutation matrices of the permutations $(k-1)(k-2)(k-3)\cdots 1k$ and $k(k-1)(k-2)\cdots 1$ are shape-Wilf equivalent.*

Theorem. *The permutation matrices corresponding to the permutations 12 and 21 are shape-equivalent for 1-alternating AD-Young diagrams.*

Theorem. *The permutation matrices corresponding to the permutations 213 and 321 are shape-equivalent for 1-alternating AD-Young diagrams.*

Theorem (Backelin-West-Xin). *For all k, the permutation matrices of the permutations $(k-1)(k-2)(k-3)\cdots 1k$ and $k(k-1)(k-2)\cdots 1$ are shape-Wilf equivalent.*

Theorem. *The permutation matrices corresponding to the permutations 12 and 21 are shape-equivalent for 1-alternating AD-Young diagrams.*

Theorem. *The permutation matrices corresponding to the permutations 213 and 321 are shape-equivalent for 1-alternating AD-Young diagrams.*

Corollary. *For all $t > 2$ and all permutations q of $\{3, 4, 5, \cdots, t\}$, the patterns $12q$ and $21q$ are equivalent for alternating permutations. For all $t > 3$ and all permutations q of $\{4, 5, 6, \cdots, t\}$, the patterns $123q$, $213q$ and $321q$ are equivalent for alternating permutations.*

Beyond Alternating Permutations

Motivation

- Joel's question in his paper.

Motivation

- Joel's question in his paper.
- Bijection from permutations to Young tableaux
  - Definition of tableau

				6	8
		3	5	7	9
1	2	4	10		

  - Entries increase left to right; top to bottom

Motivation

- Joel's question in his paper.
- Bijection from permutations to Young tableaux
  - Definition of tableau

				6	8
		3	5	7	9
1	2	4	10		

  - Entries increase left to right; top to bottom
  - l : Number of adjacent edges between adjacent rows
  - k : Number of cells per row (except top row)
  - n : Total number of cells/values in the permutation
  - Ex. (2, 4, 10); $l = 2, k = 4, n = 10$

Reading Words

				6	8
		3	5	7	9
1	2	4	10		

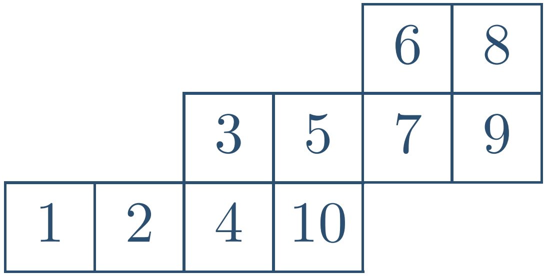
- Reading word: 124(10)357968
- Pattern avoidance is exactly as in permutations.

Reading Words

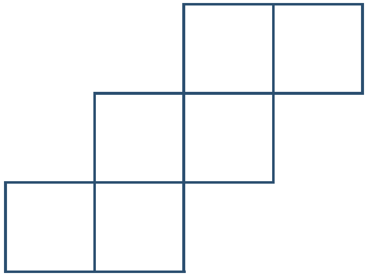
				6	8
		3	5	7	9
1	2	4	10		

- Reading word: 124(10)357968
- Pattern avoidance is exactly as in permutations.
- Define $U_n^{k,l}(r)$ to be the set of permutations p that fill tableau of the form (l, k, n) and such that p avoids r.

Reading Words



- Reading word: 124(10)357968
- Pattern avoidance is exactly as in permutations.
- Define $U_n^{k,l}(r)$ to be the set of permutations p that fill tableau of the form (l, k, n) and such that p avoids r.
- Alternating permutation pattern avoidance is a special case: $A_n(r) = U_n^{2,1}(r)$.



321 Avoidance

Theorem. *For $t > 1$, we have*

$$\left|U_{kt+1}^{k,1}(321)\right| = \sum_{i=k(t-1)+2}^{kt} \left|U_i^{k,1}(321)\right|.$$

Theorem. *For $t > 1$, we have*

$$\left|U^{k,1}_{kt+1}(321)\right| = \sum_{i=k(t-1)+2}^{kt} \left|U^{k,1}_{i}(321)\right|.$$

Example when $k = 3$:

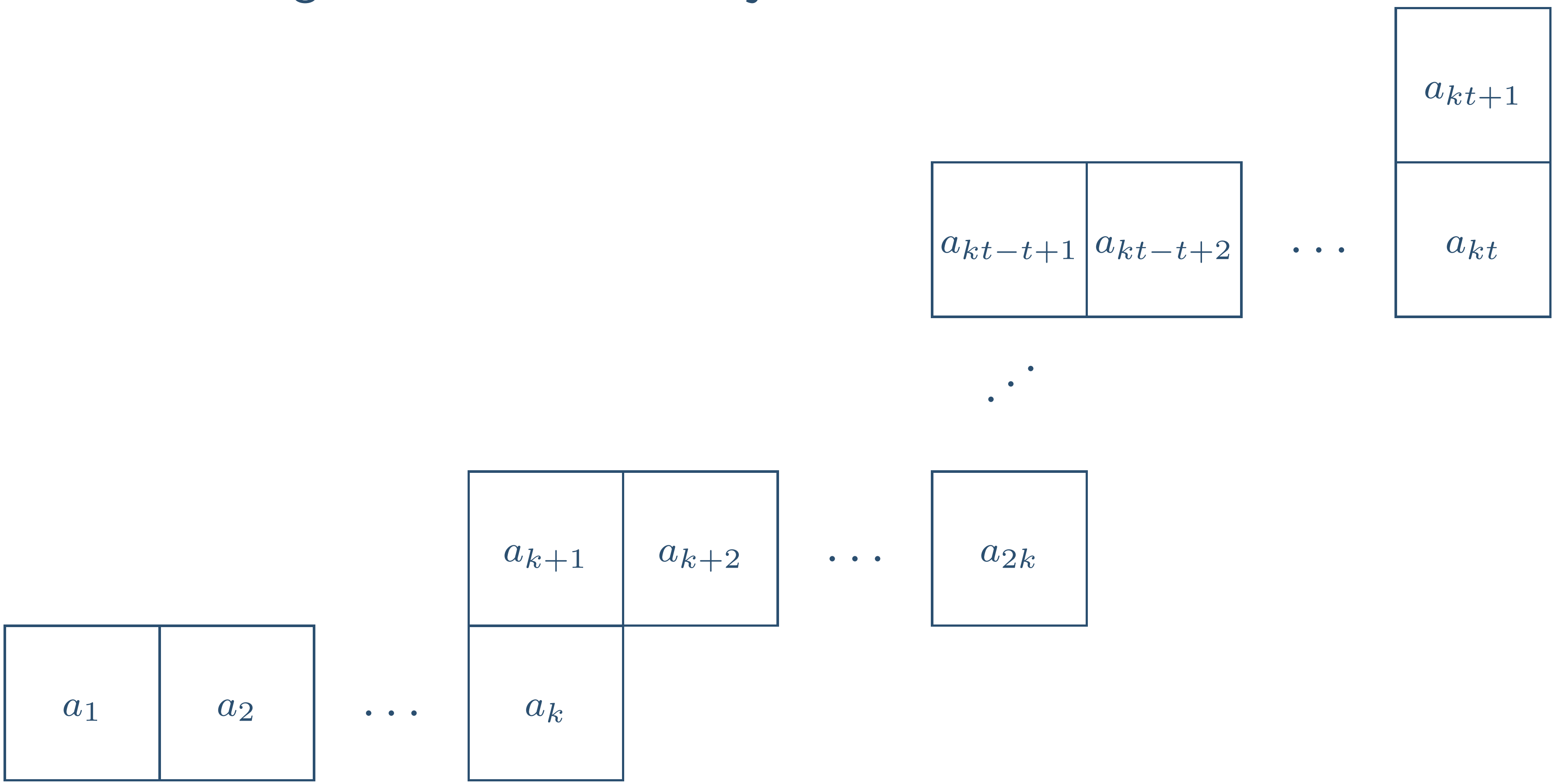
$$\left|U^{3,1}_{3t+1}(321)\right| = \left|U^{3,1}_{3t-1}(321)\right| + \left|U^{3,1}_{3t}(321)\right|$$

Some data:

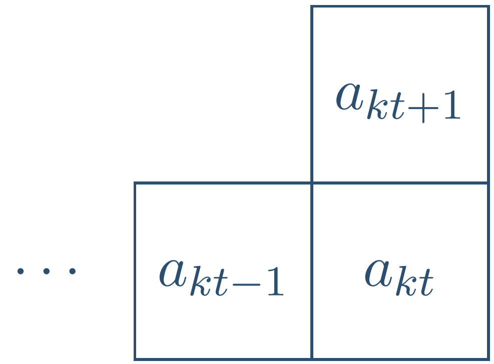
n	1	2	3	4	5	6	7	8	9	10
$\left\|U^{3,1}_{n}(321)\right\|$	1	1	1	3	9	19	28	90	207	297

Outline of Proof Regarding 321 Avoidance

$l = 1$: one edge shared between adjacent rows



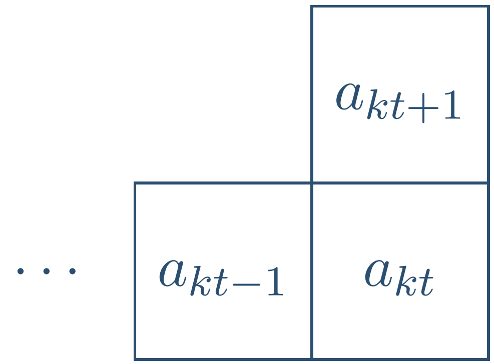
Outline of Proof Regarding 321 Avoidance



Claim: $a_{kt} = kt + 1$.

- Assume for sake of contradiction that $a_{kt} < kt + 1$.

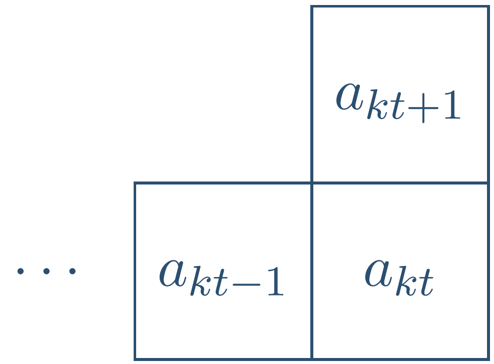
Outline of Proof Regarding 321 Avoidance



Claim: $a_{kt} = kt + 1$.

- Assume for sake of contradiction that $a_{kt} < kt + 1$.
- Since $a_{kt+1} < a_{kt}$, we have $a_{kt+1} \neq kt + 1$.

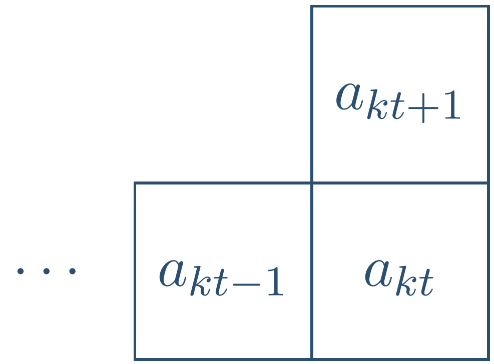
Outline of Proof Regarding 321 Avoidance



Claim: $a_{kt} = kt + 1$.

- Assume for sake of contradiction that $a_{kt} < kt + 1$.
- Since $a_{kt+1} < a_{kt}$, we have $a_{kt+1} \neq kt + 1$.
- So, for some $i < kt$, we have $a_i = kt + 1$.

Outline of Proof Regarding 321 Avoidance



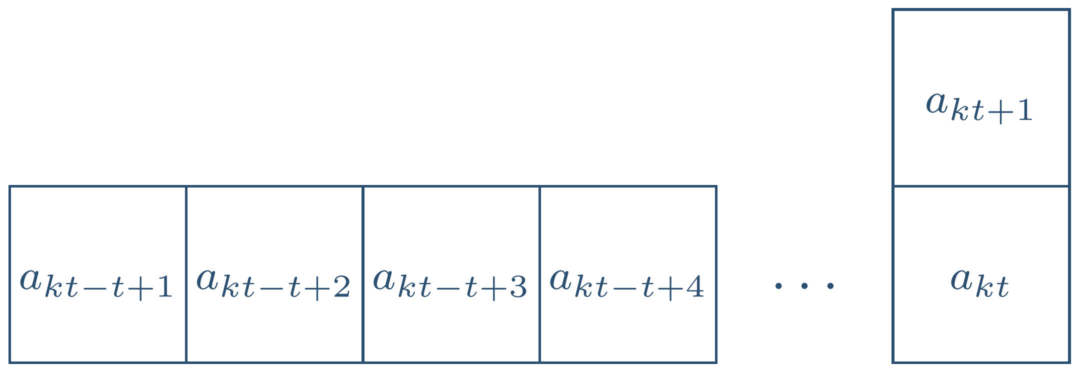
Claim: $a_{kt} = kt + 1$.

- Assume for sake of contradiction that $a_{kt} < kt + 1$.
- Since $a_{kt+1} < a_{kt}$, we have $a_{kt+1} \neq kt + 1$.
- So, for some $i < kt$, we have $a_i = kt + 1$.
- Then, $a_i a_{kt} a_{kt+1}$ is order-isomorphic to 321, contradiction.

Outline of Proof Regarding 321 Avoidance

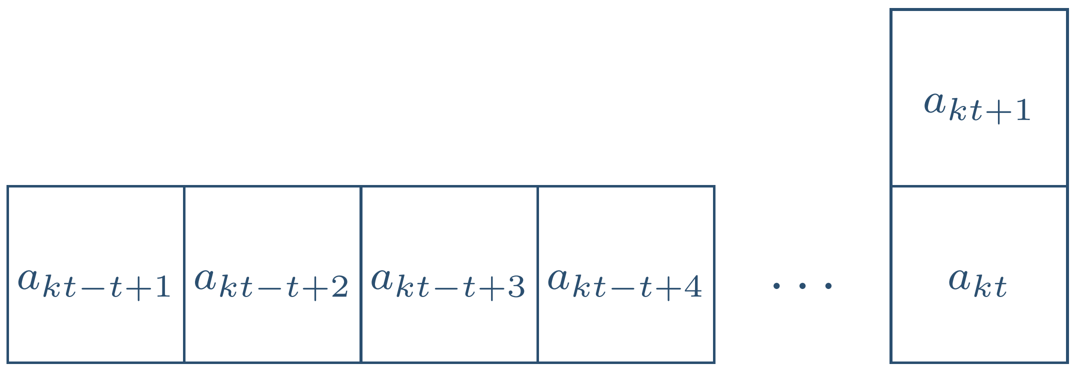
Define a *consecutive block* to be a subsequence $a_i a_{i+1} \cdots a_j$ of $a_1 a_2 \cdots a_n$, such that the values a_k are consecutive and in increasing order for $i < k < j$.

We remove the largest consecutive block with anchor (last value) a_{kt} for each permutation in $U^k_{kt+1}(321)$; suppose that the block has length s. Then,

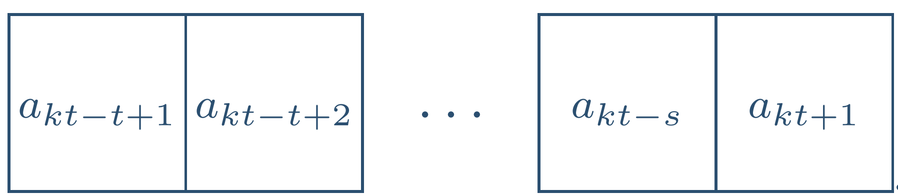


Outline of Proof Regarding 321 Avoidance

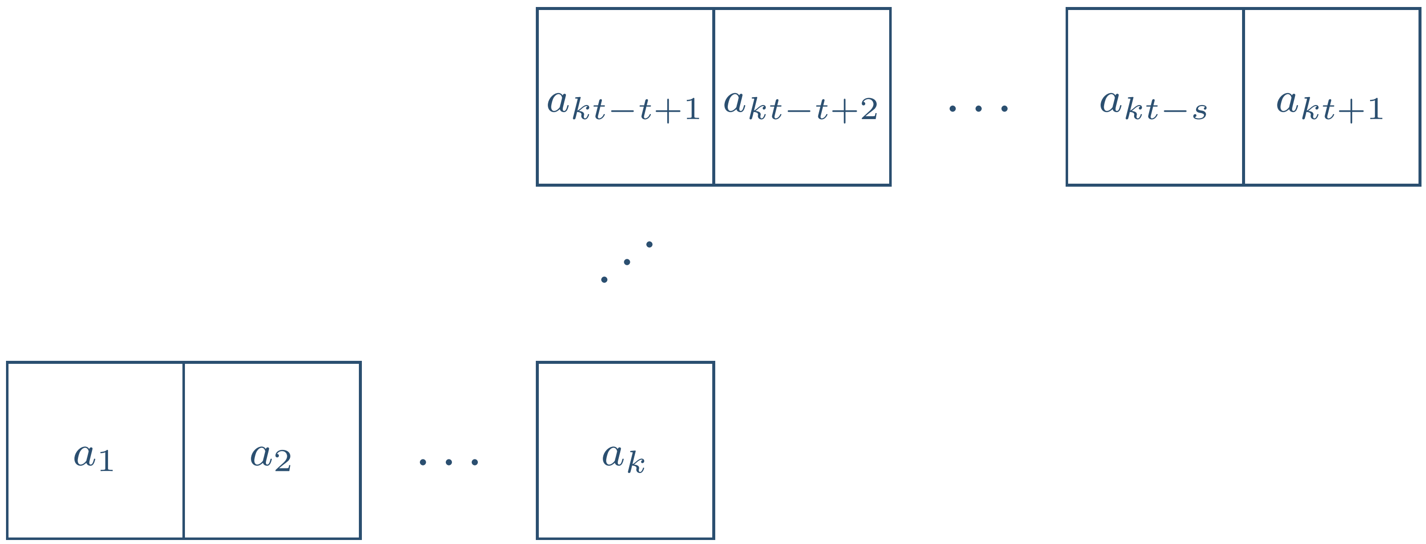
Define a *consecutive block* to be a subsequence $a_i a_{i+1} \cdots a_j$ of $a_1 a_2 \cdots a_n$, such that the values a_k are consecutive and in increasing order for $i < k < j$.
We remove the largest consecutive block with anchor (last value) a_{kt} for each permutation in $U^k_{kt+1}(321)$; suppose that the block has length s. Then,



is sent to

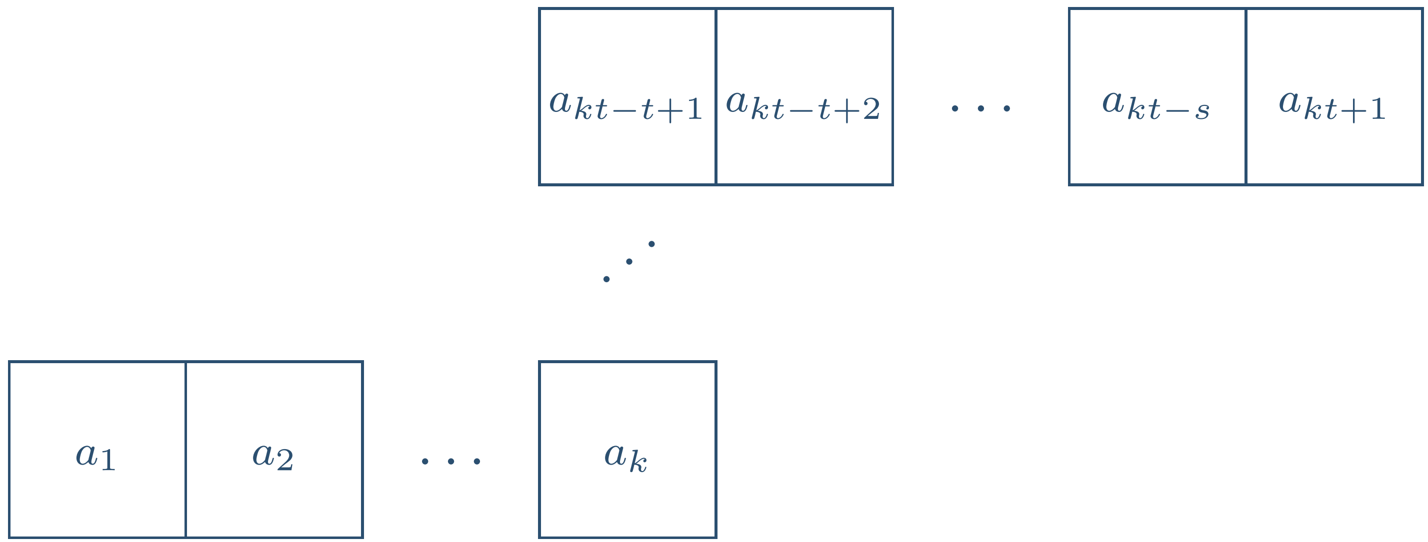


Outline of Proof Regarding 321 Avoidance



The other direction of inserting a consecutive block is clear. Thus, the bijection holds.

Outline of Proof Regarding 321 Avoidance



The other direction of inserting a consecutive block is clear. Thus, the bijection holds.

$$\left|U_{kt+1}^{k,1}(321)\right| = \sum_{i=k(t-1)+2}^{kt} \left|U_i^{k,1}(321)\right|.$$

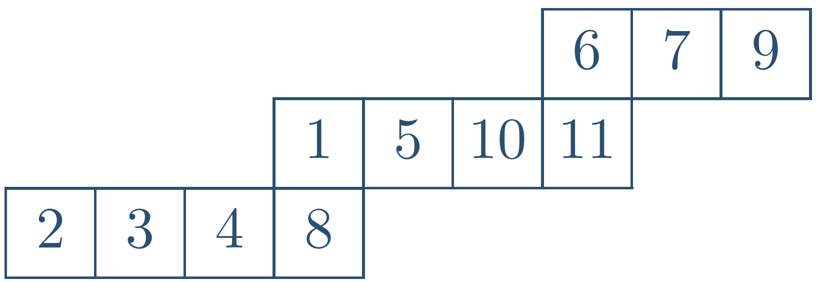
Further Application of (321)-avoidance

- This gives us a nice enumeration of $U_n^{k,l}(321)$ for $n = kt + 1$.
- What about $n = kt + m$?

Further Application of (321)-avoidance

- This gives us a nice enumeration of $U_n^{k,l}(321)$ for $n = kt + 1$.

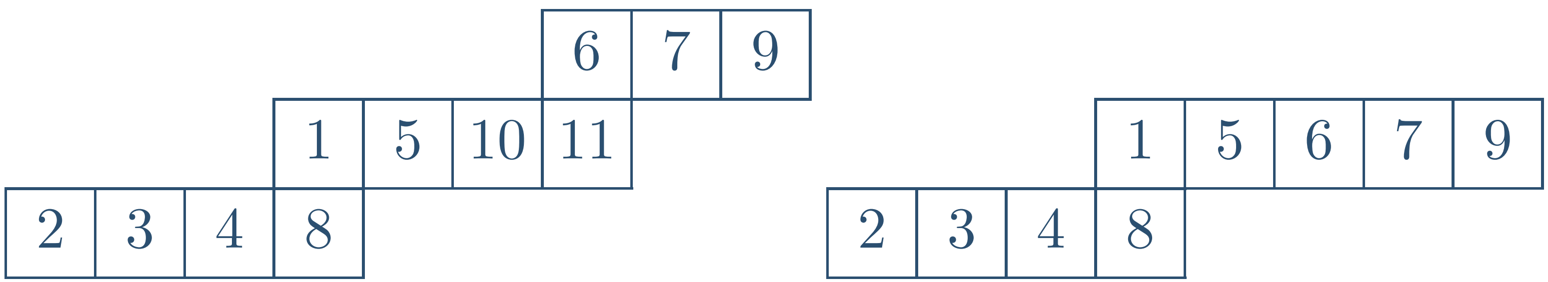
- What about $n = kt + m$?
- A similar removal of a consecutive block likely holds, but the procedure of "collapsing" the highest row into the row under it may result in a row with more than k elements:



Further Application of (321)-avoidance

- This gives us a nice enumeration of $U_n^{k,l}(321)$ for $n = kt + 1$.

- What about $n = kt + m$?
- A similar removal of a consecutive block likely holds, but the procedure of "collapsing" the highest row into the row under it may result in a row with more than k elements:



Further Application of (321)-avoidance

- This gives us a nice enumeration of $U_n^{k,l}(321)$ for $n = kt + 1$.

- What about $n = kt + m$?
- A similar removal of a consecutive block likely holds, but the procedure of "collapsing" the highest row into the row under it may result in a row with more than k elements:

						6	7	9
			1	5	10	11		
2	3	4	8					

			1	5	6	7	9
2	3	4	8				

- Thus, we will likely need to define new classes (different from $U_n^{k,l}$) to describe such tableaux, and so, the recursion for this case is likely more complicated, but not intractable.

Data for $l = 0$

- Now we turn to the $l = 0$ case.

Data for $l = 0$

- Now we turn to the $l = 0$ case.

For $k = 3$:

	1342	1243	2341	3124	2134	4123
1	1	1	1	1	1	1
2	1	1	1	1	1	1
3	1	1	1	1	1	1
4	3	3	3	4	4	4
5	6	6	6	**10**	**10**	**10**
6	10	10	10	**10**	**10**	**10**
7	37	38	38	60	60	60
8	90	94	94	**180**	**190**	**190**
9	180	190	190	**180**	**190**	**190**
10	725	806	806	1330	1400	1400

- Only avoidance patterns of a particular structure show nontrivial repetitions for $n = m$ and $n = m + 1$ for large n.
- Let q be a permutation of length t that is structurally dictated as a single down-step followed by $t - 2$ up-steps, i.e. $q = b123 \cdots (b-1)(b+1) \cdots (t-1)t$ with $b \neq 1$.
- We shall call such patterns *repetitive patterns*.

Theorem. *For $k \geq t-1$ and q a repetitive pattern, we have*

$$\left|U^{k,0}_{km+(t-2)}(q)\right| = \left|U^{k,0}_{km+(t-1)}(q)\right| = \left|U^{k,0}_{km+t}(q)\right| = \cdots = \left|U^{k,0}_{km+k}(q)\right|$$

- The approach to this is a bijective proof.
- Based on the pattern q, we perform an insertion of the proper value into a corresponding location.
- This serves as a surprising result for no other patterns contain repeats; for all other patterns q, $\left|U^{k,0}_{n}(q)\right| < \left|U^{k,0}_{n+1}(q)\right|$ (except for patterns of the form $123\cdots t$ of course).

Possible Further Directions to Our Work

- The result in the previous slide is quite nice, but it is very limited. However, checking numerical data indicates that a similar theorem holds for $l > 0$.

Acknowledgements

Thanks to

- Our mentor Joel Lewis for his valuable insight and guidance.
- The PRIMES program for making this experience possible.
- Our parents for their support.

Thanks to all of you for listening.