Modular representations of Cherednik algebras associated to symmetric groups

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Complex Reflection Groups and the Cherednik Algebra

Let $\mathfrak{h}$ be an n-dimensional complex vector space. A *reflection* is a finite-order operator s on $\mathfrak{h}$ such that $\operatorname{rank}(s - I_n) = 1$. A finite subgroup of $GL(\mathfrak{h})$ is a *complex reflection group* if it is generated by reflections.

Definition

Pick a function $c\colon G \to \mathbb{C}$ that is invariant across the conjugacy classes of G, and let $\hbar$ be a complex number. The Cherednik Algebra $H_{\hbar,c}(G,\mathfrak{h})$ is $T(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes \mathbb{C}[G]$, modulo the relations

$$[x, x'] = 0, \quad [y, y'] = 0,$$
$$[y, x] = \hbar\langle y, x\rangle - \sum_s c(s)\langle y, \alpha_s\rangle\langle \alpha_s^\vee, x\rangle s, \forall x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h}.$$

We work with $G = S_n$, and we can carry over these definitions to an algebraically closed field of characteristic p.

- "Lowest weight" representations of the Cherednik Algebras $H_{\hbar,c}(G,\mathfrak{h})$ are constructed from *Verma modules*, whose definition is motivated by the representation theory of Lie algebras.
- Let τ be a representation of G. We let $\mathrm{Sym}(\mathfrak{h})$ act as 0 on τ and construct the Verma Module

$$M_c(G,\mathfrak{h},\tau) = H_{\hbar,c}(G,\mathfrak{h}) \otimes_{\mathbb{C}[G]\ltimes \mathrm{Sym}(\mathfrak{h})} \tau.$$

- M_c has a unique maximal proper submodule J_c, and we can then construct $L_c = M_c/J_c$.
- We can study J_c as the kernel of a particular bilinear form $\beta_c\colon M_c(G,\mathfrak{h},\tau) \times M_{\overline{c}}(G,\mathfrak{h}^*,\tau^*) \to \mathbb{C}$ that has recursive properties.

- The Cherednik Algebra is $\mathbb{Z}$-graded, i.e.

$$H_{\hbar,c} = \cdots \oplus H_{-1} \oplus H_0 \oplus H_1 \oplus \cdots,$$

  where when $x \in A_m, y \in A_n$, we have $xy \in A_{m+n}$.
- The modules M_c and L_c inherit the grading from the $H_{\hbar,c}$.
- The *Hilbert series* of L_c is $\sum_{i=0}^{\infty} (\dim(L_c)_i) t^i$.
- The main goal of the project is to be able to compute Hilbert series for all $L_c(\tau)$.

Why Positive Characteristic?

- The positive characteristic case has not been well-studied, one of the reasons being the absence of general tools in dealing with it.
- As with Lie Algebras, over positive characteristic the center of a Cherednik Algebra becomes much larger. As a result, the algebra, which is very large, ends up with finite dimensional representations: $L_c(\tau)$ is finite dimensional and its Hilbert series is thus finite.
- The representation theory of S_n becomes more complicated in characteristic $p \leq n$, making relating the Cherednik Algebras to the combinatorial structure of their associated representations a more interesting problem.

- Latour studied the Cherednik algebra for $\mathbb{Z}/l$ when p does not divide l
- Katrina Evtimova studied the case when p does divide l under the direction of Emanuel Stoica.
- Martina Balagovic and Harrison Chen studied the Cherednik algebra for other groups such as $GL_n(\mathbb{F}_q)$ and $SL_n(\mathbb{F}_q)$ They determined the Hilbert series for $GL_n(\mathbb{F}_q)$ for τ trivial and all q, $n \geq 2$, also for $GL_2(\mathbb{F}_q)$ and all τ
- Unlike these, we work with groups that are examples in char. 0 reduced mod p and higher rank

Bezrukavnikov-Finkelberg-Ginzburg studied representations in the context of algebraic geometry in characteristic $p > n$ and used the fact that there is a large center

Theorem (Gordon)

The Hilbert series for $L(S_\lambda)$ when $\hbar = 0$ and p does not divide $n!$:

$$n! \prod_{s \in \lambda} \frac{1}{h(s)} \frac{1 - t^{h(s)}}{1 - t}$$

s ranges over boxes in the diagram of λ and $h(s)$ is the hook length

However, this does not work in the modular case: Gordon relied on a certain algebraic variety being nonsingular, which fails for small p

Some of our Results

Theorem

For $p > n$, $\hbar = 1$, c generic, $G = (\mathbb{Z}/m)^n \rtimes S_n$, $\underline{\lambda}$ an m-tuple of partitions, the Hilbert series for $L(S_{\underline{\lambda}})$ is

$$n! \prod_{s \in \underline{\lambda}} \frac{1}{h(s)} \frac{1 - t^{mph(s)}}{1 - t}$$

Theorem

For τ trivial, p divides n, $\hbar = 0$, Hilbert series is $\frac{1-t^p}{1-t}$ and generators of J are $x_1 - x_2, x_1 - x_3, \ldots, x_1 - x_n, x_n^p$.
For τ trivial, $\hbar = 1$, $p = 2$, and n even, Hilbert series is $(t+1)^n(t^2+1)$

$n = 5$ and $p = 3$ gives $1 + 4t + 9t^2 + 15t^3 + 16t^4 + 11t^5 + 4t^6$ (disproves conjecture that the quotients are always Gorenstein)

The data suggests the following formulas, which we are in the process of proving:

- For n odd, $p = 2$, $\hbar = 1$, c generic, the Hilbert series is

$$(t+1)^n(t^6 + (n-1)t^4 + (n-1)t^2 + 1)$$

  For $\hbar = 0$,

$$t^3 + (n-1)t^2 + (n-1)t + 1$$

- When $n = 2 \pmod 3$, $p = 3$, and $\hbar = 0$ is

$$(1+t)(1+t+t^2)(1+(n-3)t + \binom{n-2}{2}t^2 + (n-1)t^3)$$

- When $n = 1 \pmod 3$, $p = 3$, and $\hbar = 0$ is

$$(t^2+t+1)(t^2+(n-2)t+1)$$

- These last three come from conjecture on subspace arrangements on next slide

- Let X_i be the set of all $(x_1, \ldots, x_n)$ such that some $n - i$ of the coordinates are equal.
- For $n \equiv i \pmod p$ with $0 \le i \le p - 1$ and $\hbar = 0$, the data suggests that J_c is generated by symmetric functions and the ideal of X_i. L_c seems to be a complete intersection in X_i.
- We conjecture that X_i is a Cohen–Macaulay variety when $i < p$ and can prove this when $i = 1$. (Cohen–Macaulayness fails in some cases when $p \le i$)
- We also see different subspace arrangements for the groups $G(m, r, n)$. This is interesting because it means that the ideal J_c has alternative meaning which should be helpful.
- For the groups $G(2, 2, n)$, we see coordinate subspaces, and Cohen–Macaulayness follows from Stanley–Reisner theory

- We are also working with special values of $c \in \mathbb{F}_p$ for $\hbar = 1$, in general we work with generic c
- We are beginning work on general $G(m, r, n)$ (specifically $G(2, 1, n)$ and $G(2, 2, n)$). Eventually we will work on exceptional groups.
- We also plan to work with nontrivial τ

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