Infinitesimal Cherednik Algebras of $\mathfrak{gl}_2$

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Introduction

- Main object: Infinitesimal Cherednik algebras H_c.
- Questions:
  1. Algebraic structure of these algebras?
  2. Generalization of the basic theory of $\mathfrak{sl}_{n+1}$ representation?
  3. What is happening in nonzero characteristic?
- Results:
  1. Explicit computation of the center of H_c
  2. Computation of the Shapovalov form
  3. Irreducibility criterion of Verma modules
  4. Classification of irreducible finite dimensional representations

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Associative and Lie algebras

- An associative algebra is a vector space with an associative and distributive multiplication.
- A Lie algebra is a vector space with a bilinear anti-symmetric Lie bracket $[a, b]$ that satisfies the Jacoby identity:

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$$

- If A is an associative algebra, then we can define a Lie algebra structure on A by $[a, b] = ab - ba$ for $a, b \in A$.
- If $\mathfrak{g}$ is a Lie algebra, we use $\mathfrak{U}\mathfrak{g}$ to denote its universal enveloping algebra so that $[g_1, g_2] = g_1 g_2 - g_2 g_1$ for $g_1, g_2 \in \mathfrak{g}$
- Examples of Lie algebras: $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$.

Definition of Infinitesimal Cherednik Algebras

- We will abbreviate $\mathfrak{gl}_n$ by $\mathfrak{g}$.

Definition

Let V be the standard n-dimensional column representation of $\mathfrak{g}$, and V^* be the row representation, $c : V \times V^* \to \mathfrak{U}\mathfrak{g}$.
The infinitesimal Cherednik algebra H_c is defined as the quotient of $\mathfrak{U}\mathfrak{g} \ltimes T(V \oplus V^*)$ by the relations:

$$[y, x] = c(y, x),\ [x, x'] = [y, y'] = 0$$

for all $x, x' \in V^*$ and $y, y' \in V$.

Acceptable deformations c

- We are only interested in those H_c that satisfy the PBW property: $\mathrm{gr}H_c = H_0 = U(\mathfrak{g} \ltimes (V \oplus V^*))$
- Etingof, Gan, and Ginzburg proved that H_c satisfies the PBW property if and only if c is given by $\sum_{j=0}^{k} \alpha_j r_j$ where r_j is the coefficient of τ^j in the expansion of

$$(x, (1 - \tau A)^{-1} y) \det(1 - \tau A)^{-1}$$

Center is Polynomial Algebra

- Tikaradze proved that there exist $g_1, g_2 \in \mathfrak{z}(\mathfrak{U}\mathfrak{g})$ so that $\mathfrak{z}(H_c) = k[\underbrace{t_1 + g_1}_{t_1'}, \underbrace{t_2 + g_2}_{t_2'}]$
- t_1 and t_2 are generators for the center of H_0.
- $\mathfrak{z}(\mathfrak{U}\mathfrak{g})$ is a polynomial algebra in β_1 and β_2
  - $c = r_0 = (y, x)$
    $t_1' = t_1 + \beta_1$, $t_2' = t_2 + \beta_2$
  - $c = r_1 = \beta_1(y, x) + y \otimes x$
    $t_1' = t_1 + \beta_1^2 - \beta_2 - \frac{3}{2}\beta_1$, $t_2' = t_2 + \beta_1\beta_2 - \frac{3}{2}\beta_2 - \frac{1}{4}\beta_1$

s_m

Definition

Let $\gamma = \beta_1^2 - 4\beta_2 + 1$. Define $s_m = A_m(y,x) + B_m y \otimes x$, with

$$A_m = \frac{1}{2^{m+1}} \sum_{j=1}^{\lfloor \frac{m+2}{2} \rfloor} \sum_{k=0}^{j-1} \frac{4j-m-1}{2j+1} \binom{m+2}{2k+1} \binom{m+1-2k}{2j-2k-1} \beta_1^{m+2-2j} \gamma^k$$

$$B_m = \frac{1}{2^m} \sum_{j=1}^{\lfloor \frac{m+1}{2} \rfloor} \sum_{k=0}^{j-1} \binom{m+2}{2j+1} \binom{2j}{2k+1} \beta_1^{m+1-2j} \gamma^k$$

Theorem

The algebras H_c satisfy the PBW property if and only if $c = \sum_j a_j s_j$.

Examples

- $s_0 = (y, x) = r_0$
- $s_1 = \beta_1(y, x) + y \otimes x = r_1$
- $s_2 = \frac{1}{2}\left(1 + \gamma + \beta_1^2\right)(y, x) + 2\beta_1 y \otimes x = r_2 + r_0$
- $s_3 = \gamma(\beta_1 + 1)(y, x) + \frac{1}{2}\left(1 + \gamma + 5\beta_1^2\right) y \otimes x$
- $s_4 = \frac{1}{16}\left(3 + 10\gamma + 3\gamma^2 + 18\beta_1^2 + 18\gamma\beta_1^2 - 5\beta_1^4\right)(y, x) + \frac{1}{2}\beta_1\left(3 + 3\gamma + 5\beta_1^2\right) y \otimes x$

Center of H_c

Define

$$g_1'(m) = \frac{1}{2^{m+1}} \sum_{j=0}^{\lfloor \frac{m+2}{2} \rfloor} \sum_{k=0}^{j} \binom{m+2}{2j+1}\binom{2j+1}{2k+1}\beta_1^{m+1-2j}\gamma^k - \binom{m+2}{2k+1}\binom{m+1-2k}{2j-2k-1}\beta_1^{m+2-2j}\gamma^k$$

$$g_2'(m) = \frac{1}{2^{m+2}} \sum_{j=0}^{\lfloor \frac{m+2}{2} \rfloor} \sum_{k=0}^{j} \frac{m-2j-2k+1}{2k+1}\binom{m+2}{2k}\binom{m+2-2k}{2j-2k}\beta_1^{m+1-2j}\gamma^k - \frac{2j+2k-m}{2j-2k+1}\binom{m+2}{2k+1}\binom{m+1-2k}{2j-2k}\beta_1^{m+2-2j}\gamma^k$$

Center of H_c

Theorem

For a deformation $c = \sum_i a_i s_i$, *the center* $\mathfrak{z}(H_c) = k[t_1', t_2']$, *with*

$$t_1' = t_1 + \sum a_i g_1'(i)$$

$$t_2' = t_2 + \sum a_i g_2'(i)$$

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- For $c = s_0$, $t_1' = t_1 + \beta_1$
  $t_2' = t_2 - \frac{1}{4}\gamma + \frac{1}{4}\beta_1^2$
- For $c = s_1$, $t_1' = t_1 + \beta_1^2 - \beta_2 - \frac{3}{2}\beta_1$
  $t_2' = t_2 - \frac{1}{4}\beta_1\gamma + \frac{3}{8}\gamma + \frac{1}{4}\beta_1^3 - \frac{3}{8}\beta_1^2$
- For $c = s_2$, $t_1' = t_1 + \frac{1}{2}(\beta_1\gamma - \gamma + \beta_1^3 - 3\beta_1^2 + 3\beta_1)$
  $t_2' = t_2 - \frac{3}{8}\gamma - \frac{1}{16}\gamma^2 + \frac{1}{2}\gamma\beta_1 + \frac{3}{8}\beta_1^2 - \frac{1}{8}\gamma\beta_1^2 - \frac{1}{2}\beta_1^3 + \frac{3}{16}\beta_1^4$

Isomorphism with $U(\mathfrak{sl}_{n+1})$

There is an isomorphism $\phi : H_c \to U(\mathfrak{sl}_{n+1})$ for the infinitesimal Cherednik algebra of $\mathfrak{gl}_n$, when $c = a_0 s_0 + a_1 s_1$, with $a_1 \neq 0$:

$$\begin{aligned}
\phi(\alpha) &= \alpha\,, \text{ for } \alpha \in \mathfrak{sl}_n \\
\phi(y_i) &= \frac{1}{\sqrt{a_1}}\, e_{i,n+1} \\
\phi(x_i) &= \frac{1}{\sqrt{a_1}}\, e_{n+1,i} \\
\phi(\beta_1) &= \frac{1}{n+1}\left(e_{11} + e_{22} + \ldots + e_{nn} - n\, e_{n+1,n+1} - \frac{a_0}{a_1}\right)
\end{aligned}$$

Representation Theory

Definition

- A representation of an algebra A is a vector space V with an action of A defined on it.
- V is irreducible if it has no non-trivial A-invariant subspaces.

Representation theory of $\mathfrak{sl}_n$:

- Consider the triangular decomposition $\mathfrak{sl}_n$ into $\mathfrak{sl}_n = \mathfrak{n}^+ \oplus \mathfrak{n}^- \oplus \mathfrak{h}$.
- An important representation of $\mathfrak{sl}_n$ is the Verma module, generated by an eigenvector of $\mathfrak{h}$ on which $\mathfrak{n}^+$ acts by 0.
- All "nice" irreducible representations of $\mathfrak{sl}_n$ are quotients of some Verma module by its maximal submodule.

Verma Module of H_c

- Notate the matrix with one on the i-th row and j-th column by e_{ij} and the standard bases of V and V^* by y_i and x_i respectively.
- Let $U(\mathfrak{n}^+)$, $U(\mathfrak{n}^-)$, and $U(\mathfrak{h})$ be subalgebras of H_c generated by $\{e_{12}, y_1, y_2\}$, $\{e_{21}, x_1, x_2\}$, and $\{e_{11}, e_{22}\}$ respectively.
- Since the PBW property holds, we can write $H_c = U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^+)$.
- For a weight $\lambda \in \mathfrak{h}^*$, define the Verma module as

$$M(\lambda) = H_c / \{H_c \mathfrak{n}^+ + H_c(\mathfrak{h} - \lambda(\mathfrak{h}))\}$$

Shapovalov Form

- Define the Harish Chandra projection HC : $H_c \to \mathfrak{U}\mathfrak{h}$ with respect to the decomposition $H_c = \mathfrak{U}\mathfrak{h} \oplus (H_c\mathfrak{n}^+ + (\mathfrak{n}^-)H_c)$
- Define $\sigma : H_c \to H_c$ to be the anti-involution with $\sigma(e_{12}) = e_{21}$, $\sigma(y_i) = x_i$, $\sigma(e_{ii}) = e_{ii}$.

Definition

The Shapovalov form $S : H_c \times H_c \to \mathfrak{U}\mathfrak{h}$ is given by

$$S(a, b) = \mathrm{HC}(\sigma(a)\, b)$$

Its evaluation in $\lambda \in \mathfrak{h}^*$ is denoted by $S(\lambda)$.

Motivation for Shapovalov form

Theorem

1. $S(U(\mathfrak{n}^-)_\mu, U(\mathfrak{n}^-)_\nu) = 0$ *if* $\mu \neq \nu$.*
2. The Verma module is irreducible iff $\det S_\nu(\lambda) \neq 0$ *for* $\nu > 0$.

*Thus, we can restrict the Shapovalov form without losing information.

- By computing the determinant of the Shapovalov form, we can find the irreducible Verma modules.

Central elements action on Verma Module

Theorem

For a deformation $c = \sum_i a_i s_i$, t_1' acts on $M(\lambda)$ by

$$P(\lambda) = \sum_i a_i P_{i+1}(\lambda_1 + 1, \lambda_2) = \sum_i a_i \sum_{j=0}^{i+1} (\lambda_1 + 1)^j \lambda_2^{i+1-j}$$

and t_2' by

$$\sum_i a_i \left(\frac{1}{2} P_{i+1}(\lambda_1 + 1, \lambda_2) + P_{i+2}(\lambda_1 + 1, \lambda_2) - (\lambda_1 + 1)^{m+2} - \lambda_2^{m+2} \right)$$

Shapovalov Determinant

- Define the set of positive roots
  $\Delta^+ = \{\alpha_{ij} : \alpha_{ij}(h) = h_i - h_j, \forall h = \mathrm{diag}(h_1, h_2, ..., h_n) \in \mathfrak{h}\}_{i<j}$
- Define the Kostant partition function τ as $\tau(\nu) = \dim U(\mathfrak{n}^-)_{-\nu}$

Theorem

The Shapovalov form is given by

$$\det S_\nu = \prod_{\alpha \in \Delta^+} \prod_{n=1}^{\infty} (P(\lambda) - P(\lambda - n\alpha))^{\tau(\nu - n\alpha)}$$

Outline of Proof

1. If $\det S_\nu(\lambda) = 0$, $M(\lambda)$ has a highest weight vector of weight $\lambda - \mu$ for some $\mu > 0$. Since $M(\lambda - \mu)$ is embedded in $M(\lambda)$, t_1' acts on $M(\lambda)$ and $M(\lambda - \mu)$ identically. Thus, the determinant must be a product of factors of form $P(\lambda) - P(\lambda - \mu)$.
2. We then compute the highest term of the determinant, which is the product of diagonal elements of the Shapovalov form. The highest term tells us that the factors in the determinant correspond to μ being a multiple of a simple root.
3. Finally, to compute the powers of each factor, we use Jantzen's technique: we consider $M(\lambda + t\rho)$ instead of $M(\lambda)$. This t allows us to keep track of the order of zero at any λ to verify the power of each factor precisely matches that given by the formula.

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    - When we are dealing with $c = as_0$, we must use t_2' instead of t_1'
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2. We then compute the highest term of the determinant, which is the product of diagonal elements of the Shapovalov form. If we let $\alpha = (b_1, b_2)$, this product equals

$$\prod_{\alpha \in \Delta^+} \prod_{n=1}^{\infty} \left(\sum_{i=0}^{m+1} i\, b_1 \lambda_1^{i-1} \lambda_2^{m+1-i} + (m+1-i) b_2 \lambda_1^{i} \lambda_2^{m-i} \right)^{\tau(\nu - n\alpha)}$$

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When is $L(\lambda)$ finite dimensional?

Recall that $L(\lambda) = M(\lambda)/\overline{M}(\lambda)$.

- First we show that $L(\lambda)$ can be written as

$$L(\lambda) = M(\lambda)/\{H_c e_{21}^{n_1} v_\lambda + H_c x_2^{n_2} v_\lambda + H_c x_1^{n_3} v_\lambda\}_{n_3 < n_1 \text{ or } n_3 = \infty}$$

- We follow the approach of T. Chmutova. First decompose $M(\lambda)/(H_c e_{21}^{n_1} v_\lambda)$, with v_λ being the highest weight vector:

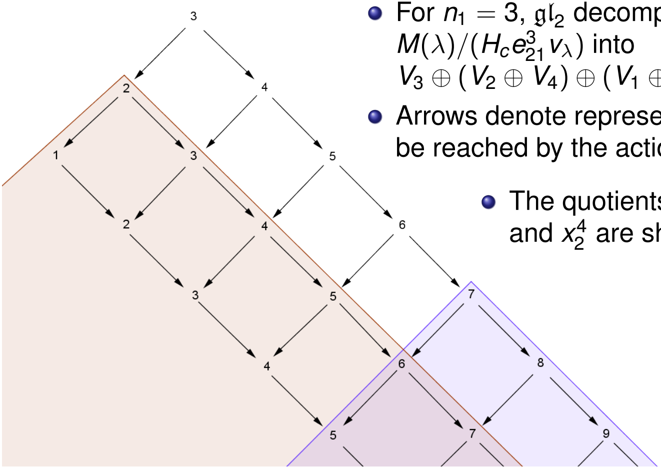
$$V_{n_1} \oplus (V_{n_1} \otimes \text{span}(x_1, x_2)) \oplus (V_{n_1} \otimes \text{span}(x_1^2, x_1 x_2, x_2^2)) \oplus ...$$

V_{n_1} is the n_1-dimensional irreducible representation of $\mathfrak{gl}_2$.

- We have $\text{span}(x_1, x_2) \cong V_2$, $\text{span}(x_1^2, x_1 x_2, x_2^2) \cong V_3$, etc
- Using Clebsh-Gordon formula, we can decompose further into

$$V_{n_1} \oplus (V_{n_1 - 1} \oplus V_{n_1 + 1}) \oplus ...$$

A Picture of $L(\lambda)$



- For $n_1 = 3$, $\mathfrak{gl}_2$ decomposes $M(\lambda)/(H_c e_{21}^3 v_\lambda)$ into $V_3 \oplus (V_2 \oplus V_4) \oplus (V_1 \oplus V_3 \oplus V_5) \oplus \ldots$
- Arrows denote representations that can be reached by the action of x_1 or x_2.
  - The quotients by x_1 and x_2^4 are shown.

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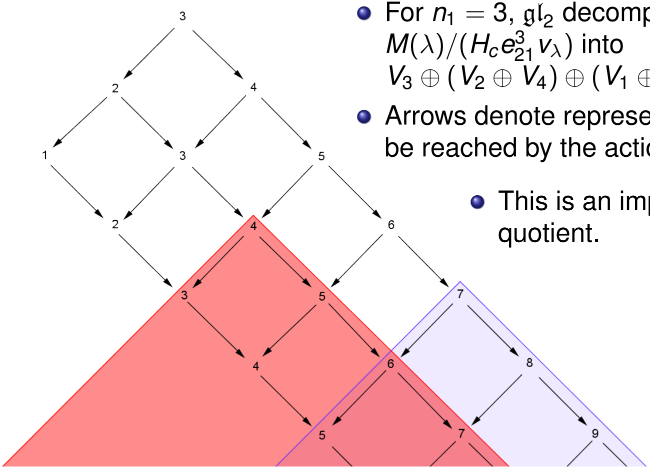

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- Arrows denote representations that can be reached by the action of x_1 or x_2.
  - This is an improper quotient.

Existance of Finite Dimensional $L(\lambda)$

- This diagram show that $L(\lambda)$ can be written as

$$L(\lambda) = M(\lambda)/\{H_c e_{21}^{n_1} v_\lambda + H_c x_2^{n_2} v_\lambda + H_c x_1^{n_3} v_\lambda\}$$

  for $n_3 < n_1$ or $n_3 = \infty$.
- We showed that there exist some λ and some c so that this $L(\lambda)$ exists.
- The dimension of $L(\lambda)$ is

$$\frac{n_2 n_3 (2n_1 + n_2 - n_3)}{2}$$

  for $n_3 < n_1$ and for $n_3 = \infty$,

$$\frac{n_2 n_1 (n_1 + n_2)}{2}$$

Summary

- We found a basis for PBW deformations and computed the central elements in this basis. We showed that the action of the center on the Verma module is particularly simple.
- We computed the Shapovalov determinant and used it to find all irreducible Verma modules and finite-dimensional $L(\lambda)$.
- Currently, we are trying to generalize our results to $\mathfrak{gl}_n$.

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