COMBINATORICS AND REPRESENTATION THEORY

VERONIKA MOROZ, ADA TSUI, ATHENA WANG

1. Introduction

The problem of expressing positive integers as sums of squares has been considered since antiquity. It is known that any positive integer is the sum of four squares, but classical proofs of this result are non-constructive. We provide an explicit decomposition of $n! = 1 \cdot 2 \cdot \ldots \cdot n$ into the sum of relatively few squares, and link this decomposition with ideas from combinatorics and representation theory, including partitions, Young tableau, and irreducible representations of the symmetric group.

More precisely, we discuss the identity

$$n! = \sum_{\lambda \vdash n} |\text{standard Young tableaux of shape } \lambda|^2$$

from two different perspectives: a combinatorial one and a representation theoretic one. The combinatorial approach uses a surprising, generalizable, and extremely useful bijection called the Robinson-Schensted correspondence, while the representation theoretic approach studies irreducible representations of the symmetric group, an accessible and interconnected topic. This paper lays out a few preliminary definitions in Section 2 before discussing the two approaches in Sections 3 and 4.

2. Definitions

First, we define a group.

Definition 1. *A group G has a set S of elements and an operation $\circ$, and respects these axioms:*

(1) Closure: if a and b are in S, then $a \circ b$ is also in S.
(2) Associativity: for any a, b, and c in S, $(ab)c = a(bc)$.
(3) Identity: there exists an identity element e in S such that for all a in G, $ae = ea = a$.
(4) Inverse: for all a in G, there exists an inverse b such that $ab = ba = e$.

We claim that the set $\mathbb{Z}$ (which is the set of integers) and the operation $+$ form a group, and the set $\mathbb{R}^*$ (which is the set of real numbers excluding 0) and the operation $\cdot$ form a group as well.

We check these alleged groups against the axioms.

For the set $\mathbb{Z}$ and the operation $+$:

(1) There is closure because the sum of any two integers is an integer.
(2) There is associativity because addition is an associate operation, so for any integers a, b, and c, $(a + b) + c = a + (b + c)$.
(3) The identity element e is 0 because for all integers a, $a+0 = 0+a = a$.
(4) The inverse of any integer a is $-a$ because $a + (-a) = (-a) + a = 0$, and $0 = e$.

Therefore, the set $\mathbb{Z}$ and the operation $+$ form a group.

For the set $\mathbb{R}^*$ and the operation $\cdot$:

(1) There is closure because for any two a, b in $\mathbb{R}^*$, $a \cdot b$ is in $\mathbb{R}^*$ as well.
(2) There is associativity because multiplication is an associative operation, so for any a, b, and c, in $\mathbb{R}^*$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
(3) The identity element e is 1 because for all a, in $\mathbb{R}^*$, $a \cdot 1 = 1 \cdot a = a$.
(4) The inverse of a is $\frac{1}{a}$ because for all a, $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1 = e$.

Therefore, the set R^* and the operation $\circ$ form a group.

(Note that 0 is specifically excluded from this set because if 0 were included in this set, then not every element of this set would have an inverse because $\frac{1}{0}$ is undefined.)

One type of groups is symmetric groups, S_n.

Definition 2. *A bijection is a mapping that is both one-to-one and onto. That is, there exactly one value in the range corresponding to each value in the domain, and one value in the domain corresponding to each value in the range.*

Definition 3. *A symmetric group S_n is a group with bijective functions $f : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ as the elements and function composition as its operation.*

This kind of bijective function is called a **permutation**.

Definition 4. *A permutation maps the elements of the set $\{1, 2, \ldots, n\}$ to the set $\{1, 2, \ldots, n\}$ in a way that is invertible.*

For example, S_3 is the set of permutations of $\{1, 2, 3\}$.

There are three ways to denote permutations: two-line notation, one-line notation, and cycle notation.

The simplest is two-line notation. On the first line, the elements of the domain are written in numerical order, and on the second line, the elements of the range that correspond to the elements of the domain are written.

For example, if 1 corresponds to 3, 2 corresponds to 5, 3 corresponds to 4, 4 corresponds to 1, 5 corresponds to 2, and 6 corresponds to 6, then the two-line notation is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 4 & 1 & 2 & 6 \end{pmatrix}$$

The first line is predictable and redundant because it is just the numbers in numerical order, so in one-line notation, the first line is deleted and only the second line is written.

Then for the previous two-line notation, the equivalent one-line notation is

$$354126.$$

Now we explain how to write this permutation in cycle notation. The first cycle starts at $1 :$ the 1 is written down, the number that 1 maps to is written down, the number that that number maps to is written down, and so on until 1 is mapped back to. That cycle is then placed in parentheses.

If the whole cycle contains the whole set, then we are done, but more likely, the cycle does not contain the whole set. Then the second cycle starts at the next smallest and available number, and so on until the cycles contain the whole set (every element of the set is in a cycle).

Then for the previous one-line notation, cycle notation is

$$(134)(25)(6).$$

This is found by first finding the first cycle. Starting at 1, it maps to 3, which maps to 4, which maps back to 1, so the first cycle is (134). The next smallest and available number is 2, so starting at 2, it maps to 5, which maps back to 2, so the second cycle is (25).

Now the next smallest and available number is 6. (Note that although $3, 4,$ and 5 are smaller than 6, they are not available because they are already in cycles.) Starting at 6, it maps back to itself, so the third cycle is (6). Now every element is in a cycle, we are done.

(Note that any cycle with only one element can be omitted, so here, (6) can be omitted and the cycle notation would be (134)(25). Therefore, if a cycle notation does not include every element, then we can assume that the unincluded elements form individual cycles and map to themselves.)

Now that we can denote permutations, we can compose permutations (also known as multiply permutations) as well. The permutations can be in one-line notation. The first permutation is written, then for each number, the number that the first permutation maps to in the second permutation is written. (Note that the numbers of the first permutation are likely not in numerical order, so this is likely not the number of the second permutation verbatim.) This can be done multiple times, and the last line is the final product.

For example, to compose 2431 and 2314, the first permutation is written

$$2431.$$

The first number of the first permutation maps to 2, and the 2 maps to 3 in the second permutation, so the first number of the product is 3. Then the second number of the first permutation maps to 4, and the 4 maps to the 4 in the second permutation, so the second number of the product is 4. And so on. Therefore, the product of 2431 and 2314 is

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

which simplifies to

$$3412$$

One example of a symmetric group are the S_n groups. Its $n!$ elements are all of the permutations of $1, 2, \ldots, n$ (these are all bijective functions), and its operation is function composition. (It is also possible to define a symmetric group on an infinite set, but we do not need this idea for the paper.)

The identity element of S_n is $12 \ldots n$.

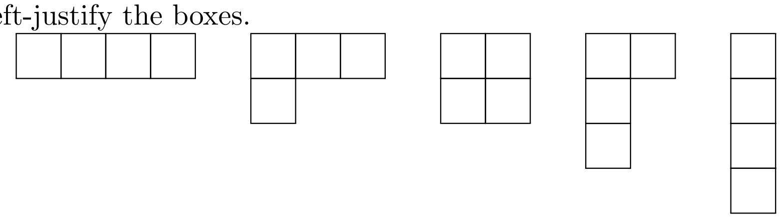
Now we define partitions.

Definition 5. *A partition $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n is a k tuple that sums to n.*

For example, the partitions of 4 are $(4), (3, 1), (2, 2), (2, 1, 1)$, and $(1, 1, 1, 1)$. These partitions can be illustrated by Young diagrams.

Definition 6. *Given a partition $(\lambda_1, \lambda_2, \ldots, \lambda_k)$, a Young diagram is a collection of $\sum_i \lambda_i$ boxes so that row i has λ_i boxes.*

Here are the partitions of 4, illustrated as Young diagrams. By convention, we left-justify the boxes.



Definition 7. *Given a partition $(\lambda_1, \lambda_2, \ldots, \lambda_k)$, a standard Young tableau is a diagram with k rows with $\lambda_1, \lambda_2, \ldots, \lambda_k$ boxes, respectively. Each box is assigned with exactly one value such that the values increase top to bottom in columns and left to right across rows.*

For example, the partition $(2, 2)$ of 4 has 2 standard Young tableaux:

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$$

And the partition $(3, 1)$ of 4 has 3 standard Young tableaux.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \multicolumn{1}{|c|}{4} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \end{array}$$

The partition $(2, 1, 1)$ of 4 also has 3 standard Young tableaux.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \multicolumn{1}{|c|}{4} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \multicolumn{1}{|c|}{4} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array}$$

Notably, each standard Young tableaux of $(2, 1, 1)$ is a standard Young tableaux of $(3, 1)$ that has been reflected over the top-left to bottom-right diagonal. In fact, if we reflect any standard Young tableaux over the top-left to bottom-right diagonal, then we get another standard Young tableaux.

3. Robinson-Schensted Correspondence

The Robinson-Schensted Correspondence is a bijection between permutations and pairs of standard Young tableaux (SYT). The two standard Young tableaux are called the Insertion Tableaux, P, and the Recording Tableaux, Q. To get a better understanding of what this means, let's look at an example. Suppose we have some permutation

$$(123)(45)$$

. First, we convert to one-line notation, and get

$$23154$$

. We use this one-line notation to create the standard Young tableaux through the Schensted Algorithm, like this:

First, take the first element, 2, and draw it in the top-left corner. Then, proceed left-to-right, inserting each new number x in the one-line notation with the following rules:

(i) If x is smaller than all the numbers already placed, put it in the top-left corner and then bump every other number in that column down one cell.
(ii) If x is bigger than the number at the rightmost end of the top row, place it to the right of the top row so it becomes the new rightmost element and the length of the top row increases.
(iii) If x is bigger than the number in the top-left corner but smaller than a different number in the top row, find the smallest number in the

top row that is bigger than x and place x in it's position. Then, bump each other number in the column down one cell. If this results in any element in any row having a gap between it and the element(s) to it's left, then move the element to the left so that no row has any gaps in it.

This method of making standard Young tableaux is well-defined because there is only one way to do it, and the Young tableaux it produces are standard. Namely, the rows are in ascending order from left to right, because of the specifications in rules (ii) and (iii). The columns are ascending from left to right because of the specifications in rules (i) and (iii).

So, for our example, after we insert the 2, we next need to insert the 3. By rule (ii), we place it at the end of the top row.

2	3

Next, we need to insert 1, so we apply rule (i).

1	3
2	

Then, we need to insert 5, so we apply rule (ii) again.

1	3	5
2		

Finally, we need to insert (4), so we apply rule (iii).

1	3	4
2	5	

We have produced one standard Young tableaux. This is the Insertion Tableaux, P. However, the Schensted Algorithm produces a pair of standard Young tableaux. To get the second standard Young tableaux, we simply invert the permutation. That is, we write a permutation such that this permutation multiplied by our original permutation yields the identity. An easy way to do this is to write each cycle backwards in cycle notation, then find the one line notation again.

Our original permutation was (123)(45), so our inverted permutation is $(321)(54) = (132)(45)$. In one-line notation, this is (31254).

Now, we repeat the same process to find the second standard Young tableaux, the Recording Tableaux, Q.

First, we insert the 3. Then, we insert the 1, using rule (i). Then, we insert the 2, using rule (iii). Then, we insert the 5, using rule (ii). Finally, we insert the 4 using rule (iii).

1	2	4
3	5	

According to the Robinson-Schensted correspondence, P and Q have the same shape and the same size.

There is another interpretation of the relationship between P and Q, which is that Q records the order in which the boxes in P are inserted, and P records the order in which the boxes in Q are inserted. This illuminates another way to construct Q given P: if you construct P, and then separately write down the order that each cell is filled in, you will also get Q. Going back to our example:

First, we filled the top-left cell, so the top-left cell is marked with 1. Then, we filled the cell to its right, which becomes marked with 2. Then, we filled the leftmost cell in the second row, which is marked 3. Then, we filled the the third cell in the top row, which is marked 4. Finally, we filled the second cell from the left in the bottom row, which is marked 5.

4. Irreducible representations of the symmetric group

Definition 8. *A general linear group is the set of all $n \times n$ matrices with a non-zero determinant under the group operation of matrix multiplication. $GL_n(\mathbb{R})$ is the general linear group with all real matrices, while $GL_n(\mathbb{C})$ denotes the matrices that have complex entries.*

Remark: $GL(n, \mathbb{R})$ is another way of denoting $GL_n(\mathbb{R})$

So, $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ would be an example of an element in $GL_2(\mathbb{R})$, while $\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right]$ would be an element of $GL(1, \mathbb{C})$.

In representation theory, the general linear group is a very important group to define. Based on Cayley's Theorem, which states that every group is isomorphic to a permutation group (a subgroup of a permutation group) and the fact that we can always express a permutation group as a matrix, we can represent every group as a general linear group.

Definition 9. *A homomorphism is a mapping $\phi : G \to H$ that preserves the group operation. Therefore, for all $g_1, g_2 \in G, \phi(g_1 g_2) = \phi(g_1)\phi(g_2)$.*

Let's look at an example of a homomorphism.

Take the group $GL_2(\mathbb{R})$. Then, we can have a map $\phi : G \to \mathbb{R}^*$ by $\phi(n) = det(g)$ where $g \in G$. Then, we can check if this is a homomorphism by using general elements g_1 and g_2 from G and seeing if it maps to an element of $\mathbb{R}^*$.

$\phi(g_1g_2) = det(g_1g_2) = det(g_1)det(g_2) = \phi(g_1)\phi(g_2)$.

Therefore, the mapping is a homomorphism.

Note: By convention, "$\mathbb{R}^*$" is used to denote the $\mathbb{R}/0$.

Now, let's talk about how we can represent some groups as matrices. To do that, we will need to use GL groups.

Definition 10. *A representation of a group G is a homomorphism $\phi : G \to GL(V)$ where GL is a general linear group over the vector space V.*

Remark: In this paper, we will only consider representations of a finite group over a finite-dimensional vector space.

Definition 11. *A character of a representation is a function χ on a general linear group such that $\chi_\rho g = Tr(\rho(g))$, where ρ is a function mapping the elements of the group to its corresponding matrix.*

Definition 12. *A dimension of a representation is the dimension of V.*

Now, let's look at an example and see how we would define each of these terms.

Let's take the map: $\phi : \mathbb{Z}/3\mathbb{Z} \to GL_1(\mathbb{R})$.

We will define our functions:
$\rho_1(g) : \begin{bmatrix} id \end{bmatrix}$ (the trivial representation)
$\rho_2(g) : e^{\frac{2\pi i}{3}g}$
$\rho_3(g) : e^{\frac{4\pi i}{3}g}$

Then, this map is a *representation* of the group G.
Character χ_1: $\chi_1(0) = 1$, $\chi_1(1)$ to 1, $\chi_1(2) = 1$.
Character χ_2: $\chi_2(0) = 1, chi_2(1) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, chi_2(2) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.
Character χ_3: $\chi_3(0) = 1, chi_3(1) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}, chi_3(2) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.
Finally, the dimension of each representation is 1, since the dimension of V is 1.

Using all of this information, we can create something called a *character table* to help us visualize this a bit easier. The top row is elements of $\mathbb{Z}/3\mathbb{Z}$ and the first column is characters.

	0	1	2
χ_1	1	1	1
χ_2	1	$-\frac{1}{2}+i\frac{\sqrt{3}}{2}$	$-\frac{1}{2}-i\frac{\sqrt{3}}{2}$
χ_3	1	$-\frac{1}{2}-i\frac{\sqrt{3}}{2}$	$-\frac{1}{2}+i\frac{\sqrt{3}}{2}$

Let's look at another example:
$\rho_1 : S_3 \to \mathrm{GL}(1, \mathbb{C})$
$\rho_2 : S_3 \to \mathrm{GL}(1, \mathbb{C})$
$\rho_3 : S_3 \to \mathrm{GL}(2, \mathbb{C})$

We will define our functions:
$\rho_1(g) = 1$ (the trivial representation)
$\rho_2(g) = sgn(g)$ (the sign representation)
$\rho_3(g)$: $e \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (1\ 2) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, (1\ 3) \mapsto \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix},$
$(2\ 3) \mapsto \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, (1\ 2\ 3) \mapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, (1\ 3\ 2) \mapsto \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$

Then, the characters would be
χ_1: $\chi_1(g) = 1$ for all $g \in S_3$.
χ_2: $\chi_2(g) = 1$ if g is even and -1 if g is odd.
χ_3: $\chi_3(e) = 2$, $\chi_3(g) = 0$ if g is a 2-cycle, $\chi_3(g) = -1$ if g is a 3-cycle.

So, our character table would be

	Conjugacy Classes		
Representations	e	(a b)	(a b c)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Now, let's define a few more terms.

Definition 13. *A reducible representation ρ is a representation V is one that can be written as the direct sum of proper representations (one that is not the original representation) where $V = V_1 \oplus V_2$. Therefore, the matrices in the image of ρ are conjugate to block matrices, and each block corresponds to a proper subrepresentation.*

Notice that we can always represent $V = V_1 \oplus V_2$ with V_1 in the upper left corner and V_2 in the lower right, with 0s all around.

$$\begin{array}{cc} \boxed{\begin{array}{l}\text{rep}\\ V_1\end{array}} & 0 \\ 0 & \boxed{\begin{array}{l}\text{rep}\\ V_2\end{array}} \end{array}$$

Definition 14. *An irreducible representation is one that is not reducible.*

Therefore, we will only use the irreducible representations for the characters. For example, for $\mathbb{Z}_2 \to GL(2, \mathbb{C})$, we would not use

$\rho(g) = \begin{bmatrix} 1 & 0 \\ 0 & (-1)^g \end{bmatrix}$.

Instead, we use $\rho_1(g) = \begin{bmatrix} 1 \end{bmatrix}$ and $\rho_2(g) = \begin{bmatrix} (-1)^g \end{bmatrix}$

Theorem 1. *There is a bijection between the irreducible representations of the group S_n and the conjugacy classes of S_n.*

Proof. Let us begin by defining Schur's lemma.

Schur's Lemma: Let V, W be irreducible representations of a finite group G over an algebraically closed field K of characteristic 0. Then, every homomorphism $V \mapsto W$ of representations is either 0 or an isomorphism. Every homomorphism $V \mapsto V$ is a multiplication by a constant.

Now, we can look at the representation $K[G]$ of the group G. We can then let $C[G] \subseteq K[G]$ generated by elements of the form $\sum_{g \in C} g$ where C is a conjugacy class of G. Then, all elements of $C[G]$ commute with all other elements of $K[G]$, so they are homomorphisms of the representations of G. We then have a K-algebra homomorphism,

$$C[G] \mapsto \bigoplus_{V irreducible} K$$

whose component that corresponds to V sends an element of $C[G]$ to its action on V, which is a scalar due to Schur's Lemma. So, if we split $K[G]$ into the irreducible representations of G, the kernel of the homomorphism is just 0, which means that the number of irreducible representations of G is at least the number of conjugacy classes of G.

To prove that the number of conjugacy classes of G is at least the number of irreducible representations of G, just consider the representation,

$Hom(V, W)$ (the vector space of all linear mappings from V to W with dimension mn).

Therefore, the character table is square. Since the irreducible representations are the rows of the character table and the conjugacy classes are the columns, the number of irreducible representations of S_n is equal to the number of conjugacy classes of S_n. $\square$

There is also a bijection between partitions and conjugacy classes since two permutations in S_n are conjugate if and only if they have the same cycle structure. There is a natural way of corresponding each irrep of S_n with a partition.

Theorem 2. *The dimension of an irreducible representation equals the number of ways to turn the corresponding partition into a standard Young tableaux.*

There is a proof of this result in Fulton and Harris' *Representation Theory.*

Definition 15. *A left regular representation can be defined as a linear map λ_g, with $g \in G$ and h being the basis vectors of the representation, such that*

$$\lambda_g : h \mapsto gh, \forall h \in G.$$

Definition 16. *An inner product of functions $G \to k$ is*

$$\langle \alpha, \beta \rangle := \frac{1}{|G|} \sum_{g \in G} \alpha(g)\overline{\beta(g)}.$$

Theorem 3. *The size of a group equals the sum of the squares of the dimensions of its irreps.*

Proof. Let G have irreps ρ_1 to ρ_r with characters $\chi_1, \chi_2, ...\chi_r$ and dimensions $d_1, d_2, ...d_4$. We want to show that $d_1^2 + d_2^2 + ...d_r^2 = |G|$.

Let ρ be a left regular representation. Let α be the character of ρ.

Because multiplication by elements that are not the identity has no fixed points other than the identity, we can say that $\alpha(identity) = |G|$ and $\alpha(g) = 0 \ \forall g \neq$ the identity. The inner product $< \alpha, \chi_i >= d_i$, because we are only multiplying the identity.

So $\alpha = \sum_{i=1}^r < \alpha, \chi_i > \chi_i = \sum_{i=1}^r d_i \chi_i$

Therefore, $\alpha(identity) = \sum_{i=1}^r d_i^2$, and $\alpha(identity) = |G|$, so $\sum_{i=1}^r d_i^2 = |G|$. $\square$

So, since the number of Standard Young Tableau that we can make is equal to the number of partitions of n, which is also equal to the number of conjugacy classes of group S_n, the number of Standard Young Tableau is equal to the number of conjugacy classes of S_n. We also have that the number of irreducible representations is equal to the number of conjugacy classes of S_n and that the size of a group equals the sum of the squares of the dimensions of its irreducible representations, which means that $n!$, which is the size of S_n would be equal to the sum of the squares of the standard young tableaux that can partition n. Therefore, we get the formula we set out to show in our introduction.

5. Acknowledgments

We are very thankful to the organizers and sponsors of MIT PRIMES-Circle for making this project possible, and send particular thanks to the PRIMES Circle coordinators, Mary Stelow and Marisa Gaetz, as well as our mentor, Katherine Tung, for their feedback on this paper and unwavering support.

Ada Tsui thanks her family, friends, and teachers, especially her mom, dad, and brother Samuel, for their constant support and endless encouragement from the very first day. Veronika Moroz thanks her family for introducing her to math (and listening to her practice presentations), and her teachers for just generally being awesome people. Athena Wang thanks her family and friends for all of their contributions and support of her math journey and sticking with her to the end.

6. References

[1] William Fulton and Joe Harris. *Representation Theory: A First Course*. Springer, 2004.

[2] Joe Gallian. *Contemporary Abstract Algebra*, 10th ed. CRC Press, 2020.

[3] David Corwin. “A Quick Proof that the Character Table of a Finite Group is a Square.” `https://math.berkeley.edu/~dcorwin/files/chartableisasq.pdf`, accessed May 21, 2023.