Stirling Numbers of the Second Kind

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MIT PRIMES Circle Spring 2023

Abstract

Suppose we had n kids and k ice cream flavors to serve the kids (each kid only getting one scoop of one flavor). How many different ways are there to give away scoops of ice cream, assuming that we use all k flavors? This question, and more, we will answer using Stirling numbers of the second kind. We will define what the Stirling numbers are, and use them to prove the "Stirling Number Dual of the Binomial Theorem".

1 Introduction

Combinatorics is a mathematical area of study concerned with counting things in a variety of different problems. On the face of it, this might sound easy, but in fact the precise study of combinatorics quickly becomes quite rigorous and challenging. Take for instance the following counting problem.

Suppose we had n kids and k ice cream flavors to serve the kids (each kid only getting one scoop of one flavor). How many different ways are there to give away scoops of ice cream, assuming that we use all k flavors? We will answer this question using what are known as Stirling numbers of the second kind.

To develop this concept, we will define permutations, combinations, and partitions in Section 2, ultimately proving the Principle of Inclusion-Exclusion (PIE). Then, we will utilize these concepts in a discussion on the binomial theorem in Section 3. Finally, we will cover how Stirling numbers of the second kind answer the above question in section 4 and also give an interesting "dual" to the binomial theorem in Section 5.

1.1 James Stirling

Before we begin solving problems with Stirling numbers involved, we give a short background behind these numbers. Firstly, these numbers are named after James Stirling, who introduced Stirling numbers in a purely algebraic setting in his book from 1730[2]. They were then rediscovered and given a combinatorial meaning by Masanobu Saka in 1782[1]. There are two different sets of numbers that bear Stirling's name: Stirling numbers of the first kind and the Stirling numbers of the second kind. For the purpose of this paper, we will focus on Stirling numbers of the second kind.

2 Introduction to Combinatorics

Combinatorics is a very important branch of mathematics that helps mathematicians count things. Combinatorics problems can get very difficult when there is no quick algorithm to help you count, and you have to find clever patterns to help you solve the problem. Mathematicians who study combinatorics have created many different operations that can help us count things in different situations. Once you can recognize when to use these tools, you can break a more challenging problem into simpler cases and solve them using these operations. In the next few sections, we will go over some of these calculation techniques.

2.1 Permutation and Combinations

A **permutation** of a set is an arrangement of its members into a sequence or linear order, or if the set is already ordered, a rearrangement of its elements. A **combination** is a selection of items from a set that

has distinct members, such that the order of selection does not matter. The difference would be that in a combination, the elements of the subset can be listed in any order. In a permutation, the elements of the subset are listed in a specific order. All data sets have a finite number of combinations as well as a finite number of permutations.
A simple example of a problem involving combinations and permutations would be:

Example 2.1. *How many ways can the numbers in "0000111" be rearranged if all repeated numbers are indistinct? [3]*

For this problem, we can use combinations, because the repeated numbers are indistinct. Since there are 3 ones and 7 possible spaces, we just have to calculate 7 choose 3.
The formula for n choose k is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

By plugging 7 in for n and 3 in for k, we get

$$\binom{7}{3} = \frac{7!}{(7-3)!3!} = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35.$$

Notice that $\binom{7}{3}$ and $\binom{7}{4}$ are the same. Thinking back to our number, we can observe this because choosing 3 spots out of the 7 to put ones in is the same as choosing 4 spots out of the 7 to put zeros in.

Example 2.2. *How many ways can the numbers in "TRAMPOLINES" be rearranged?*

Now we should use permutations, because all the letters are distinct. We have 11 letters and we want to select all 11 of those letters but in different orderings, so we do $P(11, 11)$
The formula for P(n, k) is

$$P(n,k) = \frac{n!}{(n-k)!}$$

Plugging in 11 for both n and k, we get

$$P(11,11) = \frac{11!}{(11-11)!} = 11! = 39916800$$

2.2 Partitions

Partitions are the number of ways to distribute the elements in a set with n elements into non-empty subsets, using all but not repeating any of the elements. For example, if you had the set $\{a, b, c, d\}$ and you wanted to find the partitions of it, they would be:

$$\{a\}, \{b\}, \{c\}, \{d\}$$
$$\{a\}, \{b\}, \{c, d\}$$
$$\{a\}, \{b, c\}, \{d\}$$
$$\{a, b\}, \{c\}, \{d\}$$
$$\{a\}, \{b, d\}, \{c\}$$
$$\{a, c\}, \{b\}, \{d\}$$

Each of these would be called the original set's *partitions*, *blocks*, *cells*, or *parts*. In order to make sure we have found the correct partitions, we can check the following conditions:

1. None of the partitions are empty sets
2. The union of the partitions is equal to the original set
3. The partitions are *pairwise exclusive*, meaning the intersection of any two partitions must be an empty set.

The number of partitions for a set with n elements is called the *Bell number* (notated B_n) and follows the recursion

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$$

Bell triangle is the triangle of Bell numbers where the first row is 1 and the first number in each subsequent row is the last number of the previous row. Thereafter, each number is the sum of the number to the left of it and the number diagonally above and to the left of it.

$$\begin{array}{c} 1 \\ 1 \;\; 2 \\ 2 \;\; 3 \;\; 5 \\ 5 \;\; 7 \;\; 10 \;\; 15 \\ 15 \;\; 20 \;\; 27 \;\; 37 \;\; 52 \\ 52 \;\; 67 \;\; 87 \;\; 114 \;\; 151 \;\; 203 \\ \vdots \end{array}$$

In Bell triangle, B_n is the last number in the nth row.

2.3 Principle of Inclusion-Exclusion

The **Principle of Inclusion-Exclusion (PIE)** states that the number of elements in the union of two sets is equal to the sum of the number of elements in the set subtracting the number of elements in the intersection. This can be written as

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

We can see why this is true relatively easily: "adding" the sets A and B overcounts the value of $A \cup B$. However, we can correct this overcounting by subtracting $|A \cap B|$.

In fact, the Principle of Inclusion-Exclusion generalizes to more than two sets. For instance, suppose we have three finite sets A, B, and C, and we are interested in the size of their union. To begin, we can first naively add all of their sizes together, obtaining:

$$|A| + |B| + |C|.$$

However, now we have *overcounted* the intersections of pairs of sets and now have to subtract that amount, obtaining:

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|.$$

Then, we have *undercounted* the intersection of all three and must add it back, obtaining:

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

So in total, we have that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

One can also see why this is true via the following diagram:

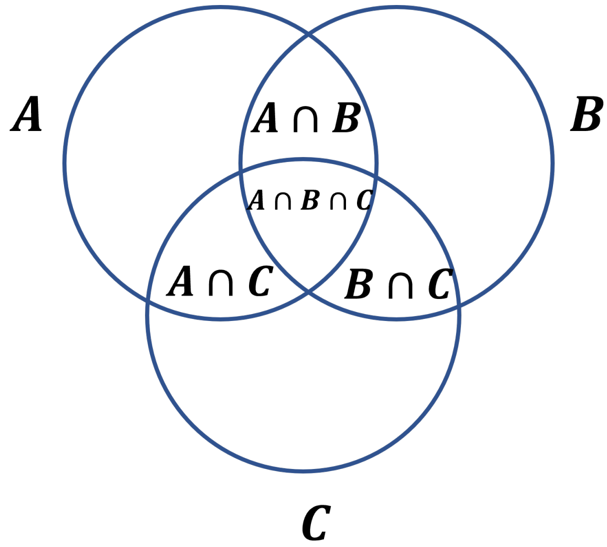

Figure 1: Venn Diagram of PIE with three sets

We can go through the same exact process for when we have n finite sets, as we see in the following proposition.

Proposition 2.3. *Given finite sets $A_1, \ldots, A_n$, one has the following identity:*

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<j\le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1}|A_1 \cap \cdots \cap A_n|.$$

The rigorous proof of this statement is fairly challenging, but the overarching idea is what we have seen so far: going through the process of overcounting and undercounting until we obtain the desired amount. This concept will become deeply useful in our understanding of Stirling numbers of the second kind.

3 The Binomial Theorem

The binomial theorem describes the algebraic expansion of powers of a binomial. It states the following.

Theorem 3.1 (The Binomial Theorem)**.** *For any $x, y \in \mathbb{R}$, we have*

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r.$$

This theorem is very useful in algebra and combintorics problems, and gives us a relatively quick way of expanding a power of a binomial expression. We prove the result in two ways: one using a combinatorial proof, and another via proof by induction.

3.1 A Combinatorial Proof of the Binomial Theorem

We first consider the expansion of a power of a binomial without simplification as an example. For instance, consider

$$(x+y)^7 = (x+y)(x+y)(x+y)(x+y)(x+y)(x+y)(x+y).$$

By multiplying out the first two factors, we get that

$$(x^2 + yx + xy + y^2)(x+y)^5.$$

If we then multiply another $(x+y)$ into the first factor of our new expression, it becomes:

$$(x^3+yx^2+xyx+y^2x+x^2y+yxy+xy^2+y^3)(x+y)^4.$$

Now, looking at how we multiplied out each factor, we find that there are two "choices" in each $(x+y)$ for each factor to be multiplied by. This means that with $(x+y)^7$, where there are seven factors, we will end up with 2^7 terms (unsimplified). Our goal is to find the coefficients of these terms *after* simplification. For example, continuing with the example of $(x+y)^7$, if we were to simplify the expression using like terms, how many terms would have x^3y^4? We can try listing these terms:

$$xxxyyyy, xxyxyyy, xxxyyxyy, \ldots.$$

These are just the combinations of the "word" XXXYYYY, of which there are $\binom{7}{4}$ (as we saw in Section 2.1. We can test this in other cases to find a pattern. For terms with x^2y^5 we find the combinations of the "word" $xxyyyyy$, which is $\binom{7}{5}$. For terms with x^7y^0 the coefficient would be $\binom{7}{0}$.

Using this, we find that we can expand $(x+y)^7$ into:

$$\binom{7}{0}x^7y^0+\binom{7}{1}x^6y^1+\binom{7}{2}x^5y^2+\cdots+\binom{7}{6}x^1y^6+\binom{7}{7}x^0y^7.$$

Now in general, when expanding $(x+y)^n$, the coefficient of the x^ky^{n-k} term in the sum is given by the combination $\binom{n}{k}$ which directly implies the binomial theorem.

3.2 An Inductive Proof of the Binomial Theorem

Our goal is to prove that:

$$(x+y)^n=\binom{n}{0}x^n+\binom{n}{1}x^{n-1}y+\binom{n}{2}x^{n-2}y^2+\cdots+\binom{n}{r}x^{n-r}y^r+\cdots+\binom{n}{n-1}xy^{n-1}+\binom{n}{n}y^n$$

First, we show that the base case of $n=1$ works

$$(x+y)^1=\binom{1}{0}x^1+\binom{1}{1}y^1$$
$$x+y=x+y$$

This holds for the base case, so we now want to show that assuming k works, $k+1$ must also work. Let k be a positive integer for which the statement is true. So

$$(x+y)^k=\binom{k}{0}x^k+\binom{k}{1}x^{k-1}y+\binom{k}{2}x^{k-2}y^2+\cdots+\binom{k}{r}x^{k-r}y^r+\cdots+\binom{k}{k-1}xy^{k-1}+\binom{k}{k}y^k$$

Next we consider the case of $n=k+1$

$$\begin{aligned}(x+y)^{k+1}=&(x+y)(x+y)^k\\ =&(x+y)(\binom{k}{0}x^k+\binom{k}{1}x^{k-1}y+\binom{k}{2}x^{k-2}y^2+\cdots+\binom{k}{r}x^{k-r}y^r+\cdots+\binom{k}{k-1}xy^{k-1}\\ +&\binom{k}{k}y^k)\end{aligned}$$

Multiplying this out, we get

$$\binom{k}{0}x^{k+1}+\binom{k}{1}x^ky+\binom{k}{2}x^{k-1}y^2+\cdots+\binom{k}{r}x^{k-r+1}y^r+\cdots+\binom{k}{k-1}x^2y^{k-1}+\binom{k}{k}xy^k+\binom{k}{0}x^ky$$
$$+\binom{k}{1}x^{k-1}y^2+\binom{k}{2}x^{k-2}y^3+\cdots+\binom{k}{r}x^{k-r}y^{r+1}+\cdots+\binom{k}{k-1}xy^k+\binom{k}{k}y^{k+1}$$

Now if we combine like terms, we get

$$x^{k+1}+\left[\binom{k}{0}+\binom{k}{1}\right]x^k y+\left[\binom{k}{1}+\binom{k}{2}\right]x^{k-1}y^2+\cdots+\left[\binom{k}{r-1}+\binom{k}{r}\right]x^{k-r+1}y^r+\cdots+\left[\binom{k}{k-1}+\binom{k}{k}\right]xy^k+y^{k+1}.$$

Using Pascal's Identity, which states that

$$\binom{n}{r-1}+\binom{n}{r}=\binom{n+1}{r}, \qquad \text{for} \quad 0<r\leq n$$

we get

$$(x+y)^{k+1}=x^{k+1}+\binom{k+1}{1}x^k y+\cdots+\binom{k+1}{r}x^{k-r+1}y^r+\cdots+\binom{k+1}{k}xy^k+y^{k+1}.$$

This is the statement we were trying to show, which proves that for any k that works, it must also be true for $k+1$. Since our base case works, this generalizes the binomial theorem for all $k \in \mathbb{R}$.

3.3 Pascal's Triangle

The binomial theorem is very closely related to Pascal's triangle. The coefficients of each term of $(a+b)^n$ according to the binomial theorem are just the numbers in the nth row.

$$\begin{array}{c}
1\\
1 \quad 1\\
1 \quad 2 \quad 1\\
1 \quad 3 \quad 3 \quad 1\\
1 \quad 4 \quad 6 \quad 4 \quad 1\\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1\\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1\\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1\\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1\\
\vdots
\end{array}$$

We can observe this pattern more clearly by listing the expansions of the first few powers of the binomial $(x+y)^n$

$$\begin{array}{c}
1\\
1x+1y\\
1x^2+2xy+1y^2\\
1x^3+3x^2y+3xy^2+1y^3\\
1x^4+4x^3y+6x^2y^2+4xy^3+1y^4\\
\vdots
\end{array}$$

Why is this true? Well, we already know that the coefficients of each term according to the binomial theorem is

$$\binom{n}{0},\binom{n}{1},\binom{n}{2},\ldots,\binom{n}{n}$$

Now we want to show that these are also the numbers each row of Pascal's triangle.

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

We can do this is to use algebra to prove that this is true:

$$\begin{aligned}
\binom{n}{r} + \binom{n}{r-1} &= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!} \\
&= \frac{n!(n-r+1) + n!(r)}{r!(n-r+1)!} = \frac{(n+1)!}{r!(n+1-r)!} \\
&= \binom{n+1}{r}
\end{aligned}$$

This shows the correlation between the numbers in the nth of Pascal's triangle correlate to the coefficients of the terms of the expansion of a binomial raised to the nth power.

4 Stirling Numbers

Stirling numbers of the second kind, notated $\left\{ n \atop k \right\}$, are the number of ways to partition n objects into k disjoint, nonempty sets.
Given the set $\{a, b, c, d\}$, if we wanted to find the number of ways to distribute the elements into 3 nonempty subsets, such as:

$$\begin{gathered}
\{a, b\}, \{c\}, \{d\} \\
\{a, c\}, \{b\}, \{d\} \\
\{a, d\}, \{b\}, \{c\} \\
\{a\}, \{b, c\}, \{d\} \\
\{a\}, \{b, d\}, \{c\} \\
\{a\}, \{b, c\}, \{d\}
\end{gathered}$$

we would obtain $\left\{ 4 \atop 3 \right\} = 6$

4.1 Stirling Number Identities

Example 4.1. *Show that for all $n > 0$,*

(a) $\left\{ n \atop 1 \right\} = 1$*,*

(b) $\left\{ n \atop 2 \right\} = 2^{n-1} - 1$*, and*

(c) $\left\{ n \atop n-1 \right\} = \binom{n}{2}$*.*

Proof.

(a) Since we only want one subset that has all of the elements, that subset must be the set itself. Since there is only one way to do this, the amount of ways to partition n numbers into 1 subset is one. □

Proof.

(b) Since we want to partition the n elements into 2 subsets, in each case, we choose how many elements we want to put in one subset, and the other must have all of the others. For example, one case is that 1 element is alone in a set, and the other $n - 1$ are all in the other set. This would be $\binom{n}{1}$. The next case would be that 2 elements are together in a set, and the other $n - 2$ are all in the other set. This would be $\binom{n}{2}$. These continue until our last case, which is when $\lfloor \frac{n}{2} \rfloor$ elements are in one set and $\lceil \frac{n}{2} \rceil$ elements are in the other. Since we know that $\binom{n}{k} = \binom{n}{n-k}$, all the cases can be written out as:

$$\frac{\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1}}{2}$$

To find the value of this, we can find the value of

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n}$$

then subtract 2 since $\binom{n}{0} + \binom{n}{n} = 2$ and divide by 2 to only get the first half.
We can find all of the combinations of n using induction. We begin with our base case of $n = 0$:

$$\binom{0}{0} = 1 = 2^0$$

This works! Now, we assume that this works for k:

$$\binom{k}{0} + \binom{k}{1} + \ldots + \binom{k}{k-1} + \binom{k}{k} = 2^k$$

and want to use this to show that if it works for k it must work for $k+1$.

$$\binom{k+1}{0} + \binom{k+1}{1} + \ldots + \binom{k+1}{k} + \binom{k+1}{k+1} = 2^{k+1}.$$

2^{k+1} is just 2×2^k, so we multiply the left side of our equation for k by 2 as well.

$$\binom{k}{0} + \binom{k}{0} + \binom{k}{1} + \binom{k}{1} + \binom{k}{2} + \ldots + \binom{k}{k-2} + \binom{k}{k-1} + \binom{k}{k-1} + \binom{k}{k} + \binom{k}{k} = 2^{k+1}$$

Now, ignoring the first and last terms, we can group each pair of terms together:

$$\binom{k}{0} + \left[\binom{k}{0} + \binom{k}{1}\right] + \left[\binom{k}{1} + \binom{k}{2}\right] + \ldots + \left[\binom{k}{k-2} + \binom{k}{k-1}\right] + \left[\binom{k}{k-1} + \binom{k}{k}\right] + \binom{k}{k} = 2^{k+1}$$

We observe a pattern and can rewrite each pair using Pascal's Identity, which states that

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

By plugging in each pair we get:

$$\binom{k}{0} + \binom{k+1}{1} + \binom{k+1}{2} + \ldots + \binom{k+1}{k-1} + \binom{k+1}{k} + \binom{k}{k} = 2^{k+1}.$$

We also know that

$$\binom{k}{0} = \binom{k+1}{0} = 1 = \binom{k}{k} = \binom{k+1}{k+1}.$$

so we can replace these in the equation.

$$\binom{k+1}{0} + \binom{k+1}{1} + \binom{k+1}{2} + \ldots + \binom{k+1}{k-1} + \binom{k+1}{k} + \binom{k+1}{k+1} = 2^{k+1}$$

This is our original equation if we plugged in $n = k+1$ so we've shown that if it works for k, it must work for $k+1$. We already know it works for our base case, so it must work for all other cases after it.
Now that we know the value of the sum of all combinations of n, we can just subtract 2 because we don't want $\binom{n}{0}$ or $\binom{n}{n}$ since those would result in an empty set. Also, because $\binom{n}{k}$ and $\binom{n}{n-k}$ are the same case, we divide our result by 2. This gives us:

$$\left\{ {n \atop 2} \right\} = \frac{2^{k+1} - 2}{2} = 2^k - 1$$

$\square$

Proof.

(c) Since there are n elements, and $n-1$ subsets, according to the *pigeonhole principle*, at least 2 must be in the same subset. Because we cannot have any empty subsets, we must have two elements in one subset and one element in every other subset. In order to choose the two elements that are in the same subset, we do $\binom{n}{2}$, so the number of different partitions of n elements into $n-1$ subsets is $\binom{n}{2}$. □

By observing Pascal's Identity, we can also propose another identity for Stirling numbers of the second kind:

$$\left\{ {n+1 \atop k} \right\} = \left\{ {n \atop k-1} \right\} + k\left\{ {n \atop k} \right\}$$

Proof. Imagine we want to split $n+1$ kids into k committees with each committee having at least one kid and each kid in exactly one committee. The total amount of ways to do this is $\left\{ {n+1 \atop k} \right\}$. We set aside one kid, we can called him James. James could either be in a committee by himself or in a committee with other kids. If he is in a committee alone, we have to put the other n kids into the other $k-1$ committees, which is $\left\{ {n \atop k-1} \right\}$. If he is a committee with other kids, we split the other n kids into all k committees and multiply this by k because there are k options for which committee to put James in. This would be $k \times \left\{ {n \atop k} \right\}$, so we get

$$\left\{ {n+1 \atop k} \right\} = \left\{ {n \atop k-1} \right\} + k\left\{ {n \atop k} \right\}$$

which proves our proposition. □

4.2 The Ice Cream Problem

Example 4.2. *Imagine that you are going to serve n kids ice cream cones, one cone per kid, and there are k different flavors available. Assuming that no flavors get mixed, show that the number of ways we can give out the cones using all k flavors is*

$$k!\left\{ {n \atop k} \right\}.$$

Proof. Arrange the n kids in a line. There are $\left\{ {n \atop k} \right\}$ ways to partition the n kids into k subsets according to the flavors that they will be receiving. After partitioning the kids, there are $k!$ ways of assigning the flavors to this partition. Hence, the of ways we can give out the cones using all k flavors is

$$k!\left\{ {n \atop k} \right\}.$$

□

This is an answer to the ice cream problem, but it is unsatisfying to us, because we still don't know how to solve for $\left\{ {n \atop k} \right\}$. In order to find a formula for Stirling numbers of the second kind, we have to solve the ice cream problem without Stirling numbers, then equate them and isolate $\left\{ {n \atop k} \right\}$.

First of all, ignoring the fact that we must use all of the flavors, there would be k^n ways to distribute the ice cream. Each kid has k choices, so we just multiply k by itself n times. Now we've overcounted, so we have to subtract the cases where we didn't use at least one flavor. The number of ways we could have done this is

$$\binom{k}{1}(k-1)^n$$

because we choose one flavor out of the total k to exclude, and then we distribute the other $k-1$ flavors. Now we've subtracted the cases where we've excluded at least two cases at least twice. We have to add back the cases where we've excluded at least two cases, which is $\binom{k}{2}(k-2)^n$ because we choice two flavors to exclude and distribute the other $k-2$ flavors to the kids. Now we have:

$$n^k - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n$$

However, now we've overcounted again. This pattern repeats as we overcount and undercount, following the Property of Inclusion-Exclusion where the sets are the cases where we exclude each of the flavors.
Our resulting expression for the number of ways to distribute the ice cream is:

$$k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n - \ldots + (-1)^k\binom{k}{k}(k-k)^n.$$

We can rewrite this as:

$$\sum_{r=0}^{k}(-1)^r\binom{k}{r}(k-r)^n.$$

Now that we have two solutions to this problem, one using Stirling numbers of the second kind, and another without them, we can set them equal to each other and isolate $\left\{ n \atop k \right\}$.

$$k!\left\{ n \atop k \right\} = \sum_{r=0}^{k}(-1)^r\binom{k}{r}(k-r)^n.$$

$$\left\{ n \atop k \right\} = \frac{1}{k!}\sum_{r=0}^{k}(-1)^r\binom{k}{r}(k-r)^n$$

5 The Stirling Number "Dual" of the Binomial Theorem

We've already established that if we were to ignore the fact that we have to use all of the flavors, we would have k^n different ways to distribute the flavors. If we were to partition these possibilities by the exact number of flavors used, letting A_1 represent the set of cases where only one flavor is used, then

$$|A_1| = \left\{ n \atop 1 \right\} \times k = k$$

because there are $\left\{ n \atop 1 \right\}$ ways to distribute the n kids into 1 subset and k ways to choose a flavor to use. We multiply these to get $\left\{ n \atop 1 \right\} \times k = k$. We can generalize that for any r

$$|A_r| = \left\{ n \atop r \right\} P(k,r)$$

because we split the n kids into r groups, where each group gets a different flavor. Then, we choose r ordered flavors out of the total k flavors to give to the groups. For positive integers r, we define

$$x^{\underline{r}} := x(x-1)(x-2)...(x-r+1)$$

Using this notation, we can write

$$|A_r| = \left\{ n \atop r \right\} k^{\underline{r}}.$$

Now we can put together all of the different possibilities for the number of flavors used to get an expression for k^n using Stirling numbers.

$$k^n = \left\{ n \atop 1 \right\} k^{\underline{1}} + \left\{ n \atop 2 \right\} k^{\underline{2}} + ... + \left\{ n \atop n \right\} k^{\underline{n}}$$

If we plug in $k = x+y$ we now have an alternative to the binomial theorem to expand $(x+y)^n$ using Stirling numbers.

$$(x+y)^n = \left\{ n \atop 1 \right\}(x+y)^{\underline{1}} + \left\{ n \atop 2 \right\}(x+y)^{\underline{2}} + ... + \left\{ n \atop n \right\}(x+y)^{\underline{n}}$$

6 Summary and Conclusion

This paper introduces Stirling numbers of the second kind and proves some important identities relating to it. It also relates two very important combinatorics operations: combinations and Stirling numbers of the second kind in order to derive a formula to calculate Stirling numbers using combinations. This opens the door to solutions for a lot more combinatorics problems because we now can plug in this formula and bridge Stirling numbers to combinations. We also applied this to find an alternative for the binomial theorem using Stirling numbers, which could be very useful in problems and proofs.

Acknowledgements

We sincerely appreciate our mentor Paige Bright's advice and help. We would also like to thank Mary Stelow and Marisa Gaetz for their comments with our paper as well as the MIT PRIMES Circle program. Cynthia would also like to thank her math teacher, Mr. Rudy Cassidy, and her parents for their encouragement and support in applying to PRIMES Circle.

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