Surfaces in Knot Theory

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Contents

1 Introduction

1.1 Knot Theory

Knot theory is the study of mathematical knots, structures that are embedded in three-dimensional space. These are not the same knots that you would see in your shoelaces or on a boat. Mathematical knots are created when the two ends of the string are permanently joined together.

> **Example**: An example of a knot is the Unknot, or just a closed loop with no crossings, similar to a circle that can be found in figure 1. Another example is the trefoil knot, which has three crossings and is a very popular knot. The trefoil knot can be found in figure 2.

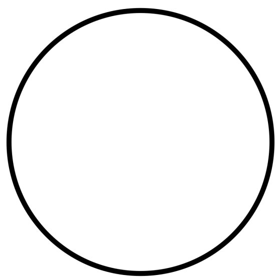

Figure 1: Unknot

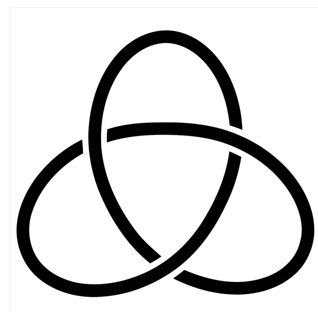

Figure 2: Trefoil Knot

Knot theory was first developed in 1771 by Alexandre-Théophile Vandermonde, while the true mathematical studies of knots began in the 19th century with Carl Friedrich Gauss. Knot theory is a branch of topology, the study of geometric properties and spatial relations unaffected by the continuous change of shape or size of figures. Knot theory has multiple real-world applications, from fluid dynamics to understanding DNA.

1.2 Reidemeister Moves

Reidemeister Moves*: transformations that change the appearance of the knot without changing its mathematical characteristics.*

There are the **twist** and **untwist**, which is when you move part of the knot over or away from itself. There are **poke** and **unpoke**, which is when you move a part of the knot over or under another part of the knot. Lastly there is **slide**, which is when you move a part of a knot over or under a crossing of the knot. These moves are shown in figure 3.

1.3 Surfaces

Let's consider three-dimensional objects and their surfaces. What is a surface? A sphere has a surface and a torus also has a surface but what does it really mean? A surface is not the shape of the sphere or the torus, but it is the outside of the geometrical figure that makes up the surface.

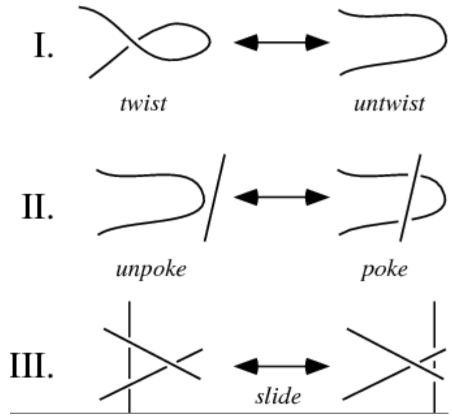

Figure 3: Reidemeister Moves

> **Example**: An example of this is if you painted the sphere a color, the surface would be the paint not the sphere itself. When we talk about surfaces, we think of them as infinitely thin, just like if we thought of paint layer to be infinitely thin.

Surfaces are important because they are a part of our everyday lives. Everything we touch and feel that is physically there has a surface. Seemingly obvious at the first glance, surfaces can get very complicated when you look at them from the math perspective. Using specific methods of topology you can prove that a sphere is the same as a cube because of the isotopy found in the surface. We will talk about that later. Knots can be found inside of the surfaces. Say you had two tori glued together as pictured in figure 4.

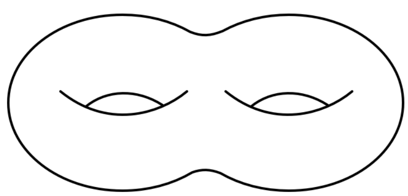

Figure 4: Double Torus

Imagine cutting the leftmost curve and the rightmost curve of the shape. Now you have two separate rods and if you shrink them down to infinitely thin you can weave and knot together to make a trefoil knot or any type of knot for that matter. After that you can just make the strings thicker depending on what knot you made and once again you will have a surface.

2 Surfaces without Boundary

2.1 Transformation and Triangulation

In order to apply surfaces to knots, we first have to determine the possibilities for surfaces. We think of all surfaces to be made of rubber, meaning that they are deformable and malleable. You can use this knowledge to state that a sphere is the same as a cube and vice versa: all you need to do is smooth out the edges if it is a cube and sharpen the edges if you start with a sphere. Treating objects like rubber is the fundamental concept behind topology. In math these rubber deformations are called an isotopy.

Isotopy: *rubber deformations of objects.*

If you start with a sphere as shown in figure 5 you can deform it to look something like a cube as shown in figure 6. By definition the two surfaces in space are still equivalent under rubber deformation and are therefore isotopic.

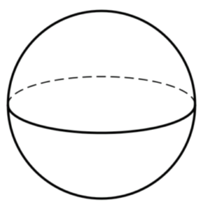

Figure 5: Sphere

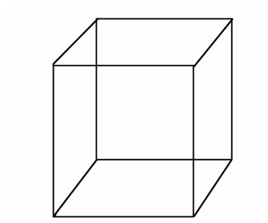

Figure 6: Cube

The property that all surfaces have in common are that at any point on the surface, there is a small disk around that point that is also part of the surface. The disk does not have to be flat, it can be deformed, but it still has to be a disk. Objects that are not surfaces are those that have neighborhoods, or blank space around a singular point. For instance if you pinched a torus to be infinitely thin then it would not be a surface anymore because the area that you pinch you could no longer make a disk around and still have every point in the disk on the surface. Even if you made the disk extremely small, it still would fail because the point is infinitely small and the region around it will not have all points on the torus.

In order to better work with surfaces we cut them up into a lot of little triangles in order to make them measurable. The triangles have to fit perfectly edge to edge so that they cover the whole surface. The triangles, like the disks, can be deformed and do not have to have straight edges. The triangles cannot overlap: they must be flush with each other. This division using triangles is called triangulation.

Triangulation: *division of a surface into triangles that fit perfectly edge to edge and cover the entire surface.*

Example: An example of triangulation is shown in figure 7 on page 5.

A question that comes up a lot is how do we know that every surface has a triangulation? This is a hard technical fact that was proven in the 1930s by Gary H. Meisters. While

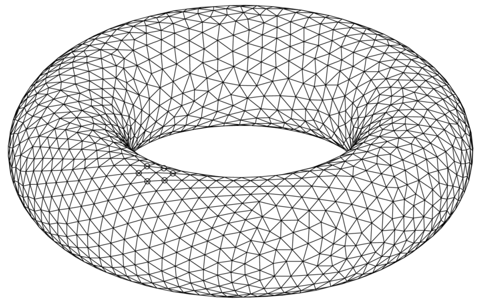

Figure 7: Triangulated Torus

every surface has a triangulation, not all surfaces have a finite number of triangles in that triangulation.

2.2 Homeomorphism

Homeomorphic Surfaces*: surfaces that are created when the the original surface triangulation is cut along the edges of some of the triangles and then glued back together along the same edges according to the instructions given by the oriented edges that were cut previously.*

This relates to knot theory because you can take a really thin torus cut through it on one side and reconfigure the string into any form of knot and as long as the edges are connected to the same edges that they were connected to before, those two surfaces are homeomorphic.

> **Example**: An example of homeomorphism of a torus and a trefoil know is shown figure 8.

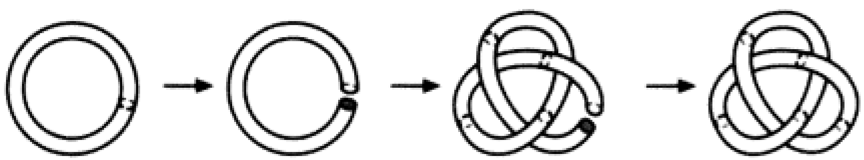

Figure 8: Example of Homeomorphism

> **Example**: An example of two surfaces that are homeomorphic but are not isotopic are a three-dimensional trefoil knot and a double torus as shown in figure 9 on page 6.

Since there is no way to change a the trefoil knot in order to end up with a double torus, the two surfaces are not isotopic. However, if you cut the trefoil knot in two places (say the rightmost loop and the leftmost loop) then if untangled you are left with two thin cylinders which now you can make into a double torus. Just keep in mind that the same edges that were connected before must be connected again in order to make the surfaces homeomorphic.

> **Example**: An example of two surfaces that are not homeomorphic would be a sphere and a torus. This is because every closed loop in a sphere must be cut into

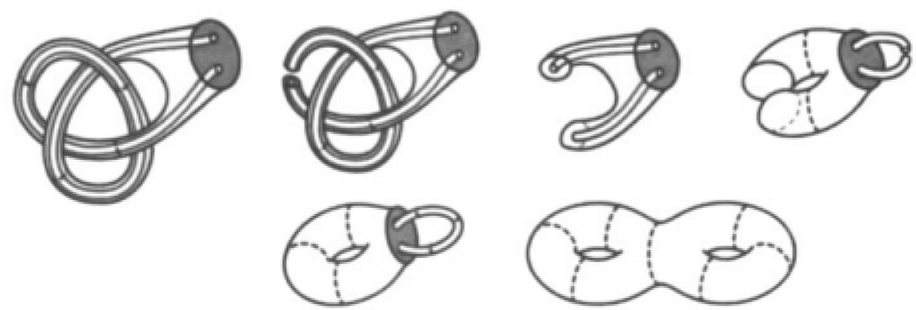

Figure 9: Trefoil Knot is Homeomorphic to a Double Torus

two pieces. While in a torus you can make a through cut on one side and you will still be left with only one cylinder meaning that a sphere is not homeomorphic to a torus.

Example: Another example of two surfaces that are not homeomorphic are a double torus and a triple torus. Both the double torus and triple torus can be found in figures 10 and 11.

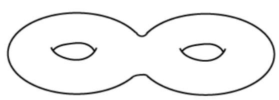

Figure 10: Double Torus

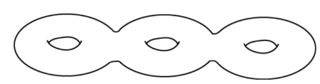

Figure 11: Triple Torus

They are not homeomorphic because if you cut the rightmost loop of the double torus and try to make the triple torus it will be impossible to reconnect the edges of the triangulation in the same order that they were in before.

2.3 Compact Surfaces and Euler Characteristic

Compact Surface*: a surface that has a triangulation with a finite number of triangles.*

Example: A torus is an example of a compact surface. Examples of surfaces that do not have a finite triangulation are a plane and a torus with a disk cut out. In the case of a plane it is obvious because the plane is infinite, while for the torus the triangles keep getting smaller and smaller as they get closer to the circle that is cut out of the torus.

Compact surfaces are of great interest to mathematicians because they have the advantage of it being possible to find their Euler Characteristic.

Euler Characteristic*: a number that describes a three-dimensional object's shape or structure regardless of how it is oriented or disfigured.*

You find the Euler characteristic by adding up the vertices (V) and faces (F) and subtracting the edges (E) of a surface in an equation: $V - E + F$.

Example: The Euler characteristic of a triangular pyramid. $V = 4, E = 6$ and $F = 4$. $4 - 6 + 4 = 2$. The Euler characteristic of a triangular pyramid is 2.

Another place that surfaces appear in knot theory is in the space around the knot. We call this the complement of the knot meaning everything except the knot itself. Let's say that X is the space that the knot is in and Y is the knot itself. We can say that $Z = X - Y$ meaning that Z is the complement of Y. It is what is left after we drill the knot out of space, and all of the surfaces we look at live in the compliment of the knot.

Every knot can be encapsulated inside of a torus. You can imagine this as just making a positively thick tube around the knot. This is illustrated in figure 12. The encapsulation can also be more unusual. You can have a torus that looks like a trefoil knot, but it can have a different number of crossings and turns inside of the torus as the original knot. This is illustrated in figure 13.

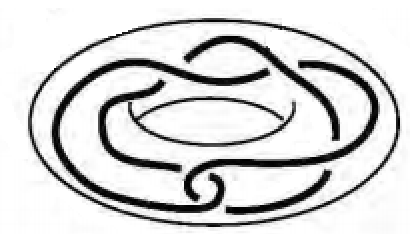

Figure 12: Knot Inside of a Torus

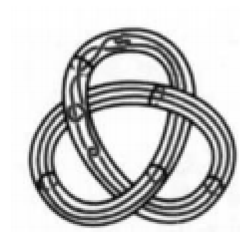

Figure 13: Knot Inside of a More Complex Figure

3 Surfaces with Boundary

3.1 Boundaries

Surface with boundary: *a surface that has at least one point where a finite triangulation does not exist.*

Boundary: *the points where there is no finite triangulation.*

An easy way to create a boundary in a surface is to remove a disk from it, as shown in figure 14.

Figure 14: Surface with Boundary

In this case we are left with a boundary in the shape of a circle. These are called boundary components. Since surfaces are thought of as made of rubber these boundary components can be deformed and do not have to be circles.

How does the Euler characteristic apply to surfaces with boundary? After removing a disk from the surface without boundary, you can think of it as removing the interior of one triangle in the surfaces triangulation. Therefore the surface's Euler Characteristic goes down by one.

> **Example**: If there are three boundary components removed from a surface then its Euler characteristic would be three less than what is was before the removal.

If you fill in a disk that has been cut out of a surface this is called **capping off** the surface.

Surfaces without boundary in three-dimensional space can be distinguished by their Euler characteristic, but surfaces with boundary cannot. There can be two surfaces with boundary which have the same Euler characteristic but are not homeomorphic, as shown in figure 15.

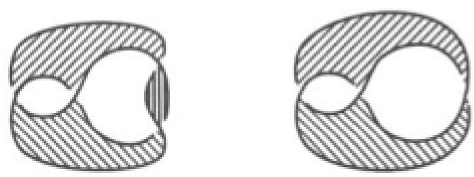

Figure 15: Same Euler Characteristic but not Homeomorphic

We can calculate the Euler characteristic of a surface with boundary the same way as before, by adding the vertices and edges to cut the surface into a finite set of faces. When we add these vertices to the boundary of the surface, you still need to count these new pieces in the calculation of the Euler characteristic.

3.2 Orientable Surfaces and Genus

You must be able to find a trait that distinguishes different surfaces and this must be a different trait than their Euler characteristic. One of these traits is that you could paint one side of a surface with boundary a color and see if you can paint all of the surface in one color. If there is still more left to be painted then it is clear that there is another side to this surface and you need to paint with a second color.

Orientable surface*: a surface that needs more than one color in order to paint the whole surface*

> **Example**: An example of an orientable surface is a torus because we could always make the outer side one color and the inner side a different color. Another example is a torus with one boundary component is orientable as well.

There are an infinite number of orientable surfaces because you could always add more tori to get a double torus, a triple torus and so on. Therefore you can always take more disks out so that there is an infinite number of orientable surfaces. Of course there are other orientable surfaces — not only tori with a boundary component are orientable.

On the other hand, a Möbius band is a one-sided surface that can be made by taking a strip of paper, giving it a half twist, then joining the ends together as shown in figure 16 on page 9.

Example: An example of a surface that is not orientable is a Möbius band.

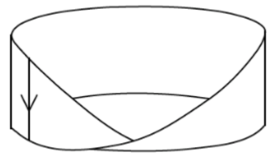

Figure 16: Möbius Band

The Möbius band is not orientable because no matter what you do there is no way to make it have more than one color necessary to paint the entire surface. We call such a surface a nonorientable surface.

Nonorientable surface*: a surface for which there is no way to make it have more than one color necessary to paint.*

Example: Another example of a nonorientable surface is a Klein bottle, as shown in figure 17.

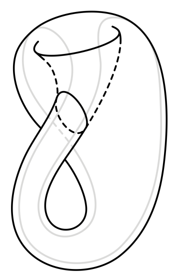

Figure 17: Klein Bottle

A Klein bottle is created by combining two Möbius bands together to create this strange three-dimensional shape that still only has one side.

Let's discuss how to find key properties of surfaces that we talked about so far. Suppose that we have a very complicated surface, shown in figure 18 where it is extremely hard to figure out whether the surface is orientable or not.

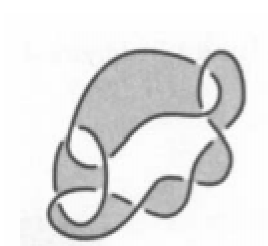

Figure 18: Complicated surface

In order to find out what kind of surface it is you need to use three specific steps.

1. Check whether is is orientable or not.

2. Look at how many boundary components it has (if there are none, then it is a surface without boundary)
3. Find its Euler characteristic.

After you have done these three steps it should be clear whether or not the surface has a boundary, and whether or not it is orientable. Also you will know its Euler characteristic which is useful for studying the surface further.

Genus: *the number of through holes that an orientable surface has.*

> **Example**: A sphere has a genus of zero because it has no holes through it, while a torus has a genus of one because it has one through hole in it.

This can go on and on by just adding more tori to create a double torus and a triple torus etc. If a surface has a boundary its genus is defined as what the genus would have been if the surface had all of its boundaries capped with disks.

Let's apply surfaces with boundary to knot theory. As the first example, let's look at an unknot. A way to define the unknot is by saying that it is the only knot that forms the boundary of a disk. In some projections of an unknot the disk may not look obvious, since the unknown may look complicated because of Reidemeister moves. You can always undo Reidemeister moves so it is still defined as an unknot, making it a disk again.

Another example of a surface with boundary in knot theory comes from composite knots.

Composite knot*: a knot that can be expresses as the composition of two knots, neither of which are a trivial knot or the unknot.*

We can think of it as having a trefoil knot and a figure eight knot and cutting them on their sides, then proceeding to connect one end from the trefoil knot to the end of the figure eight knot, and do the same for the other pair. You need to do this and make sure that there are no new unwanted crossing being created.

As shown in figure 19 there is an spherical surface surrounding the figure eight portion of the composite knot. The trefoil knot part of this image is two boundary components that lie outside of the knot.

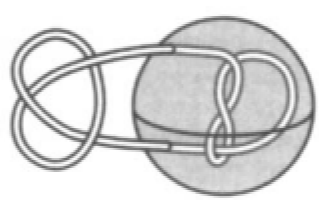

Figure 19: Composite Knot in a Sphere

Notice that for this surface you need to thicken the knot because if you left the knot infinitely thin then we would have to say that the surface is a sphere with two punctures. The punctures would occur where the knot meets the edge of the sphere. Therefore another way to define a composite knot is by saying that it is a knot such that there is a sphere in space punctured twice by the knot such that the knot is nontrivial both inside and outside of the sphere.

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