Curve-Shortening Flow

Akhil Kammila, Anshul Rastogi
Mentor: Bernardo Hernandez

June 9, 2020

Abstract

Curve-shortening flow (CSF) is a geometric heat flow with a variety of applications in mathematics and physics that acts on each point of an immersed curve inwards at a speed proportional to its curvature. In this paper, we explore the behavior of curve-shortening flow and answer questions regarding its existence and uniqueness for an arbitrary curve. We define, apply, and analyze the Gage-Hamilton, Grayson, and Huisken Theorems in the context of the curve shortening flow. Lastly, we explore special (self-similar) solutions of CSF.

1 Introduction

Curve shortening flow (CSF), aside from being a precursor to higher dimensional curvature flows, has applications in shape analysis and a connection to the well-studied heat equation. It is the one-dimensional case of the mean curvature flow, which is useful in geometry, topology, and general relativity. CSF is often studied as an introduction to complex geometric flows and for its beautiful mathematical properties.

First, we define the curve shortening flow equation as follows:

Definition 1.1. Suppose that $\gamma : [0, T) \times [0, 2\pi) \to \mathbb{R}^2$ is a family of time-dependent curves on the plane such that $\gamma(0, x)$ is an embedded curve. Denote by $k(t,x)$ and $N(t,x)$ the scalar curvature and normal vector, respectively. Then, the **curve shortening flow equation** is defined as

$$\partial_t \gamma = kN \tag{1}$$

Figure 1 illustrates the evolution of a closed curve modified by CSF. We can see that the curve's area decreases at a constant rate. This property is true for any closed embedded curve under CSF and is further discussed in section 2.1.

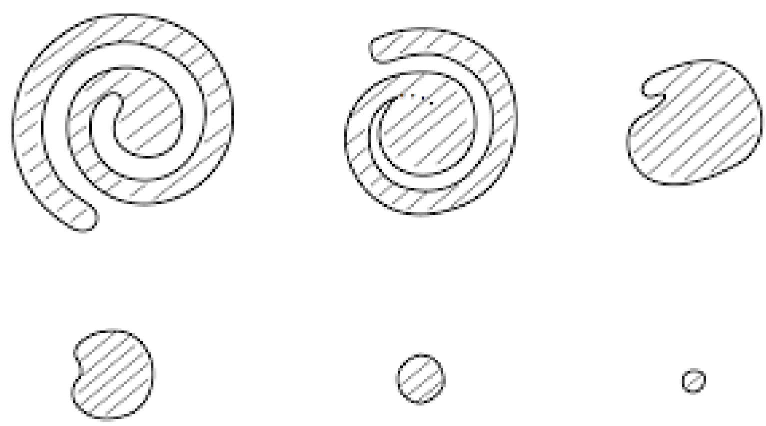

Figure 1: A closed curve undergoing CSF, progressing through time from the top left to the bottom right. Adapted from [4].

This section provides background information on curves and parabolic partial differential equations (PDEs) necessary for understanding CSF. Section 2 will define CSF and answer questions regarding its existence and uniqueness. It will also define and explain the Gage-Hamilton, Grayson, and Huisken Theorems in relation to CSF. Finally, we explore special solutions of CSF, including the grim reaper, paperclip and spiral curves.

1.1 Differential Geometry of Curves

We define $I \subseteq \mathbb{R}$ as an open interval. We then commence by establishing the following:

Definition 1.2. A **curve** is a differentiable function $\alpha : I \to \mathbb{R}^2$ from I on $\mathbb{R}$ into $\mathbb{R}^2$.

Definition 1.3. We define a **closed curve** as one that has identical endpoints at which it is smooth.

Definition 1.4. An **embedded curve** is a curve that does not self-intersect and is a bijection of an interval $I \subseteq \mathbb{R} \to \mathbb{R}^2$ onto its image.

From these definitions, any closed embedded curve must split the plane into exactly two areas.

Definition 1.5. Immersed curves are curves with a non-zero derivative. Hence, they may admit self-intersections.

If $\alpha : I \to \mathbb{R}^2$ is a curve with $\alpha = (\alpha_1, \alpha_2)$, the velocity vector of α at $t \in I$ is the tangent vector

$$\alpha'(t) = \left(\frac{d\alpha_1}{dt}(t), \frac{d\alpha_2}{dt}(t) \right) \tag{2}$$

at the point $\alpha(t)$.

We define a the **Frenet-Serret frame field** on a curve in two dimensions, a moving frame dependent upon the direction of movement of the curve at a given point, comprised of the tangent and normal vectors.

To facilitate this, we will consider reparametrizing the curve to a unit-speed curve for our initial definition.

Reparametrizing a curve entails altering the mapping of the interval I of the original curve such that the curve is preserved geometrically, but changing the speed at which we travel over the curve. This allows us to travel along curves differently to, as we will do here, enable ease when considering specific properties of the curve.

More explicitly, if $\alpha : I \to \mathbb{R}^2$ is the original parametrization and $\beta : J \to \mathbb{R}^2$ is a re-parametrization, there exists an $h : J \to I$ that is a smooth bijection and $\beta(s) = \alpha(h(s))$. Therefore $\beta'(s) = h'(s)\alpha'(h(s))$.

A **unit-speed curve** is a reparametrization of a curve such that its speed $|\alpha'|$ (the magnitude of the velocity vector $\alpha'(t)$) is always equivalent to 1. Any arbitrary curve can be reparametrized to a unit-speed curve. This is also known as the arclength reparametrization.

We will first define the Frenet-Serret frame for unit speed curves before generalizing to arbitrary-speed curves.

Definition 1.6. For a unit-speed curve $\beta : I \to \mathbb{R}^2$, the **Frenet-Serret frame** is comprised of the two orthogonal vector fields: T (the tangent vector) and N (the normal vector). As we can see depicted in Figure 2 below, the tangent vector is representative of the derivative of a given point on the curve. The normal vector is defined as the unique vector orthogonal to tangent vector at a given point.

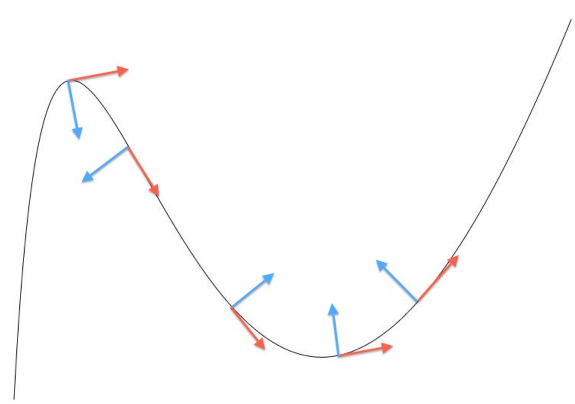

Figure 2: Here, the tangent vectors are red and the normal vectors are blue. Adapted from [1].

We define k as the **curvature** of a curve. Then, where T denotes the unit tangent vector and N denotes the principal normal vector,

$$N = T'/k \tag{3}$$

k can be visualized more intuitively as the measure of the "twisting" of a curve inwards or outwards at a given point. To see this one can show that the radius of the largest inner circle that touches the curve at only one point $\alpha(t)$ is $1/k(t)$. This means that the larger the curvature the smaller the radius (because it 'twists' more we can only fit smaller circles) and a curvature of zero implies it is a circle of infinite radius, a.k.a. a straight line.

The following proposition is a useful tool for calculating the tangent and normal vectors with arbitrary parametrizations.

Proposition 1.2. The Frenet Frame can be found for an arbitrary speed curve by normalizing each vector field as follows:

$$T = \alpha' / \|\alpha'\| \tag{4}$$

$$k = \|\alpha' \times \alpha''\| / \|\alpha'\|^3 \tag{5}$$

1.3 Parabolic Theory of PDEs

The parabolic theory of partial differential equations has a wide variety of applications in physics and engineering. For our purposes, because CSF is a parabolic equation, the theory from this section can be applied to curve shortening flow. The most prototypical parabolic PDE is the heat equation or diffusion equation. It is utilized to describe the conduction of heat in different situations.

The **heat equation** in 3-dimensions derives from the following example from physics [5]. Suppose we had a body $V \subset \mathbb{R}^3$ with heat $H(t)$ at any given time $t \in [0, T)$; let $u(x,t)$ represent the amount of 'heat energy' at spacial point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ at time t so that

$$H(t) = \int_V u(x,t)\, dV. \tag{6}$$

We can assume that the rate of change of heat of the body V is proportional to the rate of heat flowing out of the boundary. The heat flow along the surface is given by $\nabla u = \langle \partial_{x_1} u, \partial_{x_2} u, \partial_{x_3} u \rangle$. Denoting by n the normal vector to the surface of V (∂V), the flow out of the boundary is given by the integral along the surface of V of $\nabla u \cdot n$. This gives us that

$$\frac{d}{dt}H(t) = \int_V u_t(x,t)\, dV \tag{7}$$

$$= \int_{\partial V} \nabla u \cdot n\, dS \tag{8}$$

$$= \int_V (\partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2) u\, dV. \tag{9}$$

In the final step, we make use of the Divergence Theorem. The heat equation then follows by setting the first integral equal to the third and dropping the integration symbol since both are being integrated over V. We then obtain the following definition:

Definition 1.7 (Heat Equation)**.** The n-dimensional heat equation on a set $V \subset \mathbb{R}^n$ for time $[0,T)$ is given by

$$\partial_t u - \left(\partial_{x_1}^2 + \ldots + \partial_{x_n}^2\right) u = 0. \tag{10}$$

The heat equation, and more general parabolic equations, have properties that are useful in understanding CSF. These are the strong and weak maximum and minimum principles and uniqueness, a corollary of which is the strong separation principle for CSF.

The following theorems are adapted from [5]. We consider the Heat Equation problem on the area $U \subset \mathbb{R}^2$ for a time interval $[0,T]$. Now we need the following series of definitions:

Definition 1.8 (Parabolic Cylinder)**.** We define the domain of $u(x,t)$ as the **parabolic cylinder** $U_T := U \times (0,T]$. We denote the boundary of the spacial domain ∂U. Closely related is the **closed parabolic cylinder** $\bar{U}_T = U \times [0,T] \cup \partial U \times [0,T]$.

Definition 1.9 (Parabolic Boundary)**.** The **parabolic boundary** is given by $\partial U_T := U \times \{t = 0\} \cup \partial U \times (0, T)$.

Finally, denote by $C_1^2(U)$ the set of functions defined on $U \subset \mathbb{R}^2$ that are twice-differentiable in space and once-differentiable in time. This means that $\partial_{x_1}^2 u$, $\partial_{x_2}^2 u$, $\partial_{x_1}\partial_{x_2} u$ and $\partial_t u$ are all continuous functions.

Theorem 1.4 (Weak/Strong maximum principle)**.** *Suppose $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ satisfies the heat equation within U.*

1. *Then,*
$$\max_{\bar{U}_T} u = \max_{\partial U_T} u \tag{11}$$

2. *Additionally, if U is connected and there exists a point $(x_0, t_0) \in U_T$ such that $u(x_0, t_0) = \max_{\bar{U}_T} u$ then*
$$u \textit{ is constant within } \bar{U}_{t_0} \tag{12}$$

Theorem 1.5 (Uniqueness)**.** *Let $g \in C(\partial U), f \in C(U)$. Then there exists at most one solution $u \in C^2(U) \cap C(\bar{U})$ of the boundary value problem*

$$\begin{cases} u_t - \Delta u = f \text{ in } U \\ u = g \text{ on } \partial U \end{cases} \tag{13}$$

Proof. If u and $\tilde{u}$ both satisfy equation 13, apply Theorem 1.4 to the function $w := \pm(u - \tilde{u})$ □

Intuitively, the term kN on the right hand side of the CSF equation can be rewritten in terms of arclength derivatives as

$$kN = \partial_s^2 \gamma \tag{14}$$

so that the CSF equation can be rewritten as

$$\partial_t \gamma - \partial_s^2 \gamma = 0 \tag{15}$$

which looks strikingly similar to equation 10. However CSF does not preserve arclength (as will be seen in section 2), so equation 15 conceals nonlinearities that provide the interesting geometric behaviour.

Now we require a general definition of parabolicity for a geometric flow:

$$\partial_t \gamma = F(\gamma, \theta, k)n, (p, t) \in I \times (0, T), T > 0 \tag{16}$$

Here, $F = F(x, y, \theta, q)$ is a function given in $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$. We can say that F is parabolic in a set E if $\dfrac{\partial F}{\partial q}(x, y, \theta, q) > 0$ for all (x, y, θ, q) in E. Furthermore, we may say that F is uniformly parabolic if there exist two positive numbers such that

$$\lambda \leq \frac{\partial F}{\partial q} \leq \Lambda. \tag{17}$$

In the case of CSF, we have that $F(\gamma, \theta, k) = k$ e.g. $F(x, y, \theta, q) = q$. So, CSF satisfies the condition for uniform parabolicity.

The properties of the heat equation generalize to solutions of general parabolic PDE. Demonstrating this is rather extensive and is explained in detail in [5]. In particular, Theorem 1.5 shows that, given an arbitrary embedded curve, we can construct a unique CSF with it as an initial curve.

Theorem 1.4 implies something much more interesting. Suppose γ_0 and ρ_0 are embedded curves *that do not intersect*. Using Theorem 1.5, we can construct two unique curve shortening flows, γ and ρ, on $[0, T) \times [0, 2\pi)$ with $\gamma(0, x) = \gamma_0$ and $\rho(0, x) = \rho_0$. Now the question is, is there a time $t_0 > 0$ such that $\gamma(t_0, \cdot)$ intersects $\rho(t_0, \cdot)$?

The answer, surprisingly, is always no. The reason for this is that it would violate the strong maximum principle. The distance between both curves can be shown to satisfy a parabolic PDE because of CSF, so if there was a point x_0 where the distance was 0 at a time $t_0 > 0$, we get from Theorem 1.4 that the distance would be 0 from the beginning. Therefore two curves undergoing CSF can only intersect each other if they did so at the beginning of the flow.

2 Curve Shortening Flow

In this section we discuss how solutions to the CSF problem (especially closed curves and self-similar solutions) behave over time under curve-shortening flow. We also explore how the Grayson, Gage-Hamilton, and Huisken Theorems pertain to this.

2.1 Evolution of Geometric Characteristics

Curves that undergo curve-shortening flow have a variety of interesting geometric behaviors over time. Notable among them are the Grayson's and the Gage-Hamilton Theorems, which will be discussed in more detail in the following subsection (2.2).

Arclength has an interesting evolution over time. Denoting $L(t)$ as the arclength of a curve as a function of time, we deduce the relationship as follows:

We know $L(t)$ satisfies

$$\frac{\partial L}{\partial t} = \int_I \frac{\partial s}{\partial t}\, dp \tag{18}$$

From here,

$$\int_I \frac{\partial s}{\partial t}\, dp = -\int (k^2)\, ds \tag{19}$$

$$= -\int (k^2)\, ds \tag{20}$$

We see from this that the arclength of a curve is strictly decreasing over time, hence the name curve-shortening flow.

Another rather intriguing geometric property of the evolution for a closed embedded curve undergoing CSF is the behavior of the encapsulated area over time. We derive this evolution as such:

We restate the definition of γ as the CSF-produced family of curves as delineated in Definition 1.1. Denoting s as the arclength parametrization, we define the area of an embedded closed curve as

$$A = -\frac{1}{2}\int_{\gamma} \langle \gamma, N\rangle \, ds \tag{21}$$

From here, we have

$$\begin{aligned}\frac{dA}{dt} &= -\frac{1}{2}\int k\, ds + \frac{1}{2}\int (k_s)\langle \gamma, N\rangle \, ds + \frac{1}{2}\int (k^2)\langle \gamma, N\rangle \, ds &(22)\\ &= -\int_{\gamma} k\, ds &(23)\\ &= -2\pi &(24)\end{aligned}$$

Hence, we see that the area of a closed embedded curve decreases at a rate of 2π for every unit of time that the curve is under CSF. This leads to an interesting conclusion. Defining $A(t)$ as the enclosed area of a closed curve as a function of time t, we can conclude that a closed embedded curve becomes extinct at time $\dfrac{A(0)}{2\pi}$.

2.2 Grayson's and Huisken's Theorems

We now explore the Gage-Hamilton and Grayson's Theorems in the context of CSF, which define the behavior of all closed and embedded curves undergoing CSF.

Theorem 2.3 (Gage-Hamilton Theorem)**.** Consider the Initial Value Problem for the curve shortening flow, where γ_0 is a convex, embedded closed curve. A_0 is the area enclosed by γ_0 and $\omega = A_0/2\pi$. The curve γ_0 has a unique solution $\gamma(\cdot, t)$, which is uniformly convex for each t in $(0, \omega)$ As $t \uparrow \omega$, $\gamma(\cdot, t)$ shrinks to a point [3].

Theorem 2.4 (Grayson's Theorem)**.** Convergence to a round point is given by the existence of a unique point $x_0 \in \mathbb{R}^2$ such that the rescaled flows

$$\gamma_t^{\lambda} := \lambda \cdot (\gamma_{T+\lambda^{-2}t} - x_0) \tag{25}$$

converge for $\lambda \to \infty$ to the round shrinking circle $\{\partial B_{\sqrt{-2t}}\}_{t\in(-\infty,0)}$

If $\gamma : I \to \mathbb{R}^2$ is a closed embedded curve, then the curve shortening flow $\gamma_{t t\in[0,T)}$ with $\gamma_0 = \gamma$ exists until $t = \omega$. As $t \to \omega$, it converges to a round point.

Gage-Hamilton's Theorem and Grayson's Theorem combine to prove that any closed embedded curve shrinks to a single round point under curve shortening flow. Gage-Hamilton proved convergence to a point for convex and embedded curves. Grayson generalized this by proving that non-convex closed embedded curves eventually become convex, allowing the Gage-Hamilton theorem to then be applied.

2.5 Special Solutions

There are several "special" solutions to curve-shortening flow that can be explored. We see that these are self-similar and oftentimes preserve several geometric properties when undergoing CSF.

We shall commence by discussing **travelling curves**. These are solutions of CSF which take the following form:

$$v(x,t) = v(x) + ct \tag{26}$$

The curve is translated upwards with time under curve-shortening flow. During this transformation, its shape remains constant. Interestingly, there is only one travelling curve, which is known as the "grim reaper." Its equation is given by

$$y = -\log\cos(x), x \in (-\pi/2, \pi/2) \tag{27}$$

Figure 2.5 below depicts the curve and its transformation upwards with time.

Two other special solutions of CSF are derived from a combination of upwards and downwards translating grip reapers. First, the paperclip is

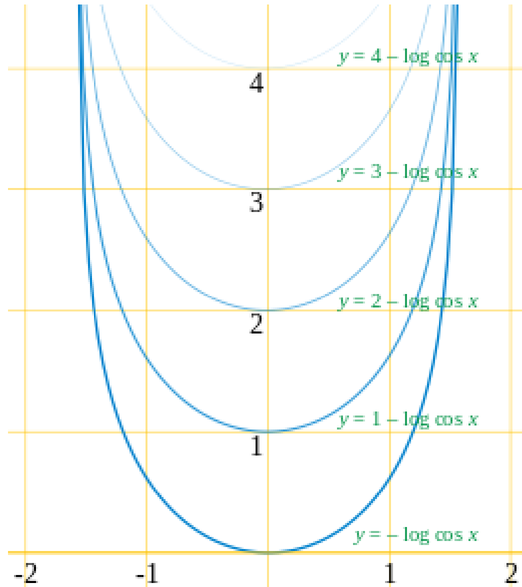

Figure 3: Grim Reaper
Adapted from [9]

defined as

$$\cosh(y(t)) = e^{-t}\cos(x(t)) \tag{28}$$

when restricted to $\|x\| < \pi/2$ This describes a shape that converges to a single point as $t \to 0$. As $t \to -\infty$, the curve becomes an oval consisting of two "grim reaper" curves joined at the ends. Figure 3 shows the paperclip curve from $t \to -\infty$ to $t \to 0$

The hairclip solution is defined as

$$\sinh(y(t)) = e^{-t}\cos x(t) \tag{29}$$

As $t \to -\infty$ the curve appears to be grim reapers alternating between translating up and translating down connected in a row. As $t \to 0$, the curve becomes a horizontal line.

Finally, we introduce **spirals**. These are translating waves that rotate around the origin about the polar angle α. They can be expressed by the

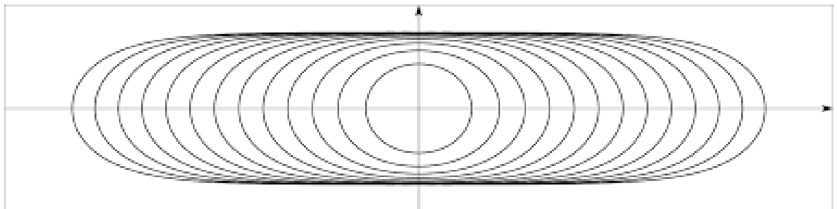

Figure 4: Paper Clip
Adapted from [2]

equation:

$$r\cos(\alpha(r) + ct), r\sin(\alpha(r) + ct) \tag{30}$$

They rotate with the constant speed $\|c\|$.

Spirals can either simply rotate, rotate while shrinking, or rotate while expanding. In order to remain self-similar under CSF, they must extend infinitely.

The yin-yang curve is a famous solution to CSF. It is a symmetric spiral which has an inflection point that remains at the origin as it is transformed.

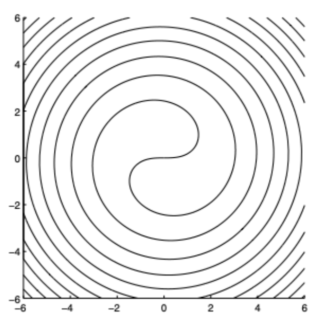

Figure 5: Yin Yang
Adapted from [6]

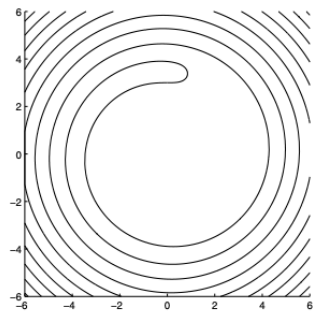

Figure 6: Rotating Spiral
Adapted from [6]

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