PRIMES CIRCLE FINAL PROJECT

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1. Introduction

In this paper, we explore combinatorial game theory. Combinatorial game theory is the study of games like Chess or Checkers, where two players alternate turns until one wins the game. We will explore some basic definitions and terms of game theory through two original games: Traveling and Beads. In section 2, we lay out preliminary concepts of game theory, which we will implement in sections 3 and 4.

2. Preliminaries

2.1. **Definitions.** In this section, we will begin by introducing formal definitions that will help us transform these game boards into strategies for winning.

Definition 2.1. ([1, Definition 1.5]) A *combinatorial game* is a 2-player game played between Louise and Richard. The game consists of the following:

(1) A set of possible *positions*, or the states of the game.
(2) A *move rule* indicating for each position what positions Louise can move to and what positions Richard can move to.
(3) A *win rule* indicating a set of *terminal positions* where the game ends. Each terminal position has an associated *outcome*, either Louise wins and Richard loses, Louise loses and Richard wins, or it is a draw.

Example 2.2. In the game Tic, there is a 1×3 array. To move, Louise marks an empty square with a $\circ$ and Richard with a $\times$. If either player gets two adjacent squares marked with his or her symbol, then they win. Tic is clearly an example of a combinatorial game as it satisfies all three conditions.

Now, we consider the condition for winning. Games can be classified by their conditions for winning.

Definition 2.3. ([1, p. 3]) A *normal-play* game is a combinatorial game such that the last person to make a move wins.

Example 2.4. Checkers is an example of a normal-play games. Both games played in this paper are normal-play games.

Games can also be categorized into the types of moves made by each player.

Definition 2.5. ([1, p. 26]) An *impartial game* is a normal-play game where the available moves from every position are always the same for either player.

Example 2.6. Tic is an example of an impartial game because the available moves are the same for Louise and Richard.

Definition 2.7. ([1, p. 26]) A *partizan game* is a normal-play game where the available moves for Louise and Richard are not the same.

Example 2.8. An example of a partizan game is Cut-Cake. In Cut-Cake there is a rectangular cake with horizontal and vertical lines running along the cake indicating where it can be cut. Richard can only make horizontal cuts and Louise can only make vertical cuts. From each position in Cut-Cake, Richard and Louise have different available moves, so the game is partizan.

Definition 2.9. ([1, p. 7]) A *strategy* is a set of decisions indicating which move to make at each position where that player has a choice.

Definition 2.10. ([1, p. 7]) A *winning strategy* is a strategy that guarantees a win for the player who follows it.

In order to expedite the process of finding a winning strategy, it is important to classify and label positions.

Definition 2.11. Position Types

(1) **Type L**: Louise has a winning strategy no matter who goes first.
(2) **Type R**: Richard has a winning strategy no matter who goes first.
(3) **Type N**: The Next player has a winning strategy.
(4) **Type P**: The Previous or second player has a winning strategy.

3. Traveling

In this section we introduce an original normal-play partizan combinatorial game, Traveling. The game is played between Louise and Richard on a $m \times n$ grid. A token is located on the top left corner of the grid. On Louise's turn, she can move the token to the left or down as many edges as she wants, staying on the grid. On Richard's turn, he can move to the right or up as many edges he wants, staying on the grid. Neither player can move the token to a previously taken position. The first player to move the token to the bottom right corner wins. For any game of Traveling, Louise wants to force Richard to reach the rightmost edge

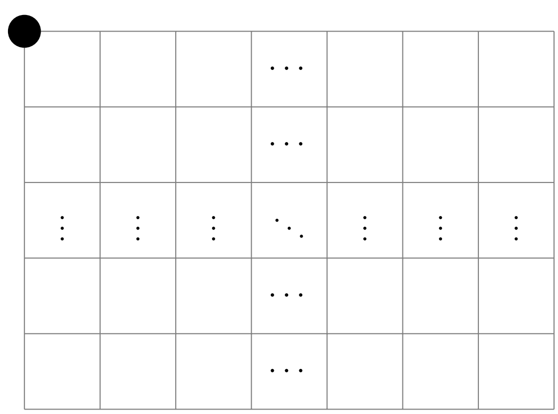
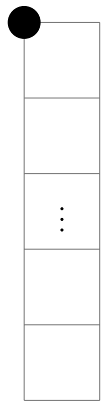


Figure 1. A starting position in an $m \times n$ game of Traveling

so she can go straight down, while Richard wants to force Louise to the bottommost edge. In essence, they are racing against each other to force the other player to their respective edge while staying as far away from their own edge as possible. For this reason, Louise does not want to move downwards and Richard does not want to move rightwards unless they have no other choice or unless it is essential to their winning strategy.

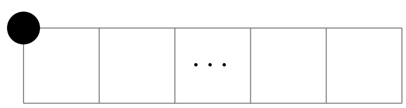
Now, we begin with some basic theorems of Traveling.

Theorem 3.1. *Any $n \times 1$ game of Traveling, where $n > 1$, is type L.*



Proof. If Louise goes first, she is only able to go any number of edges down. Her winning strategy is to go down one, forcing Richard to move to the right and thus open a downward path to the bottom right corner for Louise. If Richard goes first, again, he is only able to go to the right, enabling Louise to go straight down to the bottom right corner. Therefore, an $n \times 1$ game of Traveling is always type L. □

Theorem 3.2. *Any $1 \times n$ game of Traveling, where $n > 1$, is type R.*



Proof. This follows much of the same logic as the proof for Theorem 3.1. Now, playing optimally, Richard can move one to the right, forcing Louise to move down and creating a rightward path to the bottom right corner for Richard. If Louise goes first, she goes down, and Richard goes right to reach the corner. Thus, a $1 \times n$ game of Traveling is always type R. □

The similarity between the analysis of the $1 \times n$ and $n \times 1$ games of Traveling reflects a more general symmetry between $m \times n$ and $n \times m$ games.

Theorem 3.3. *For positive integers m and n, the following hold:*

(1) If an $m \times n$ game of Traveling is type P, then an $n \times m$ game is also type P.
(2) If an $m \times n$ game of Traveling is type N, then an $n \times m$ game is also type N.
(3) If an $m \times n$ game of Traveling is type L, then an $n \times m$ game is type R.
(4) If an $m \times n$ game of Traveling is type R, then an $n \times m$ game is type L.

Proof. We first establish the symmetry between Richard and Louise's moves. Consider a 2×1 game of Traveling vs. a 1×2 game.

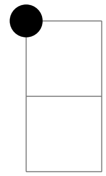


FIGURE 2. A 2×1 game of Traveling

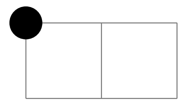


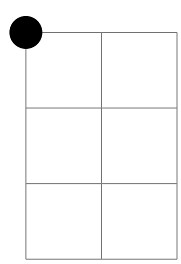
FIGURE 3. A 1×2 game of Traveling

On the 2×1 game, Richard can only move one to the right, allowing Louise to go straight down. In contrast, on the 1×2 game, Louise can only move one down, allowing Richard to go straight to the right. From here, we see that Richard moving rightward on an $m \times n$ game corresponds to Louise moving down on an $n \times m$ game. It follows that this is also true for Richard moving up and Louise moving leftward.

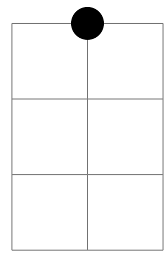
Now, consider an $m \times n$ game that is type P. If Louise goes first, that means Richard has a winning strategy for that game. Thus, for an $n \times m$ game, if Richard goes first, then Louise can follow Richard's winning strategy using the correspondence we established and have a winning strategy as well. Similarly, if Richard goes first on the $m \times n$ game, then he has a winning strategy on the $n \times m$ game. Therefore, (1) is true. This logic also applies to (2), where if an $m \times n$ game is type N and Louise goes first, then Richard can go first and thus has a winning strategy for an $n \times m$ game.

This symmetry also explains (3). If an $m \times n$ game is type L, then Louise has a winning strategy no matter what. Then, on an $n \times m$ game, Richard can follow Louise's winning strategy using the correspondence to get a winning strategy of his own, making the $n \times m$ game type R. The proof of (4) follows. $\square$

Proposition 3.1. *A 3×2 game of Traveling (and thus, a 2×3 game) is type P.*



Proof. Let Louise move first. Louise's first move must be downwards. To minimize her downward motion, she moves one edge down. From there, Richard can only move to the right, also moving only one edge, and Louise must move down one again. Now, Richard has two options: moving right one or up twice. If he moves right once, he reaches the rightmost



edge first, allowing Louise to win. Thus, he moves up twice, putting the token in the above position.

Louise has no choice but to move down three edges, putting her at the bottommost edge of the grid and allowing Richard to win. In this game, there was little choice involved except for Louise's decision to move the token downwards once for her first move.

Consider the case where Louise moves the token down twice first instead. Richard can move up one, which forces Louise to go down twice, putting her at the bottommost edge and once again allowing Richard to win.

If Louise moves the token down three edges, she automatically loses, so no matter what, Louise will lose if she goes first.

If Richard goes first, he must move to the right, and Louise must move down from there. Louise can move down one, two, or three edges, and since she wants to move down as little as possible, we consider the one-edge case. From there Richard can only move to the right, putting the token at the rightmost edge and allowing Louise to win. Thus, the 3×2 game is type P, and subsequently, so is a 2×3 game (by Theorem 3.3(a)).

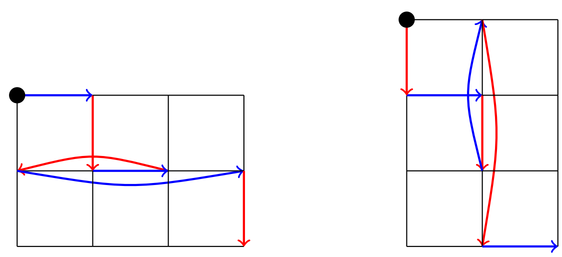
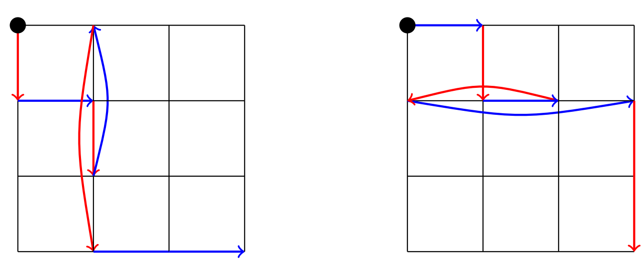


FIGURE 4. Winning strategies for Louise and Richard

□

Proposition 3.2. *A* 3×3 *game of Traveling is type P.*

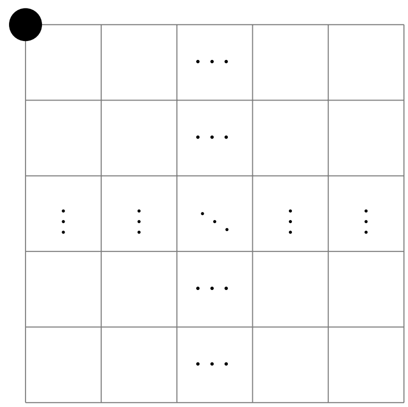


Proof. Let Louise move first. Again, she must move down, and we consider the case where she only moves down one edge. Richard must move right, presumably once, forcing Louise down one again. Richard then is able to win by moving up twice, leaving Louise no option but to move down three edges and allowing Richard a rightward path towards the bottom right corner.

If Louise moves down two edges, then Richard goes right, forcing Louise to the bottommost edge. Thus, she is never able to win if she goes first.

If Richard goes first, we see something similar. While we have established that an $m \times n$ game is symmetric to a $n \times m$ grid, an $m \times m$ game is symmetric to itself. Louise uses her corresponding version of Richard's winning strategy, and Richard cannot win. Thus, if Louise goes first on a $m \times m$ game and is unable to win, then the same is true if Richard goes first, and the game is type P. □

Corollary 3.3. *An $n \times n$ game of Traveling is either type N or type P.*



Proof. Assume for the sake of contradiction that an $n \times n$ game is type L. Then Louise has a winning strategy if she goes first. Because there is symmetry between the players' respective moves, Richard can imitate Louise's path if he goes first, which gives him a winning strategy. Thus, the game is type N, which is a contradiction. If we assume again that an $n \times n$ game is type L, Louise has a winning strategy if she goes second. Richard imitates Louise's path if he goes second, which also gives him a winning strategy. Then the game is type P, which is again a contradiction. Therefore, any $n \times n$ game of Traveling is type N or P. □

We have seen how $n \times n$ games are type N or P, while $1 \times n$ and $n \times 1$ games are type R or L. It seems to follow that in an $m \times n$ game of Traveling, the farther apart m and n are, the more likely it is that the game is type L or R.

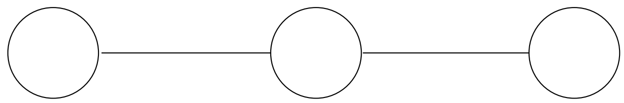
Conjecture 3.4. *For any $m \times n$ game of Traveling, if $m \geq n + 2$, the game is type L. if $n \geq m + 2$, the game is type R. This excludes the cases in Theorems 3.1 and 3.2 where we have 2×1 and 1×2 games.*

While we have not solved Traveling completely, the results included here and other examples we have explored gives us some ideas about general good practices. When Louise plays, she wants to go down as slowly as she can, choosing to go left whenever possible. The same goes for Richard, who wants to move to the right as slowly as he can. The resulting race between Louise and Richard makes a game of Traveling.

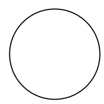
4. Beads

This normal-play, impartial game is played by removing beads from a chain. Both players have the same set of moves. Players can choose to remove 1 or 2 beads from the chain each turn, or if the number of beads on the chain is even, they can choose to split the chain into 2 smaller chains of equal length.

Example 4.1. Consider the following example with 3 beads.



In this example, Louise will go first. She has the choice of removing 1 or 2 beads. She cannot split the chain in half because there is an odd number of beads. Louise chooses to remove 2 beads.



Richard takes the single bead and wins the game.

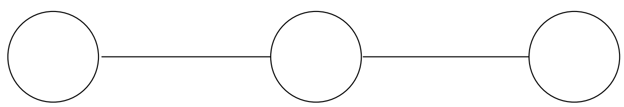


Figure 5. A chain with 3 beads

Proposition 4.2. *A 3-bead chain is Type P.*

Proof. We know both players are playing to win. In Example 4.1 we know that if Louise goes first, Richard will win. If we reverse the roles and have Richard go first, Louise will win. Therefore, the chain with 3 beads is Type P. □

Now we will cover a lemma that will be helpful in the further examples as well as the general strategy. In further examples, we will refer to this lemma as the Mirror Lemma.

Lemma 4.3. *A position of the form $\alpha + \alpha$ in an impartial game is Type P.*

Proof. α is Type N or P. Consider a position composed of two 2-bead chains. A 2-bead chain is Type N, so this position can be written as Type N + Type N. If Player 1 removes 1 bead off of a chain, then Player 2 will mirror that on the other chain. Then there will be two chains that are 1 bead long. Because players can only remove beads off of one chain per round, Player 1 will have to leave the last bead for Player 2 to take, winning the game. If Player 1 removes 2 beads off of one of the chains, then Player 2 can mirror that, also winning the game. Therefore, the position $\alpha + \alpha$ when α is Type N is a Type P position.

□

Proposition 4.4. *In this next example we will look at a chain with six beads.*

Proof. In this case, we will have Louise go first again. In order for Louise to win this game, she must be the next player to split the chain into two smaller chains. This creates two chains of 3 beads, which we can call α and α'.

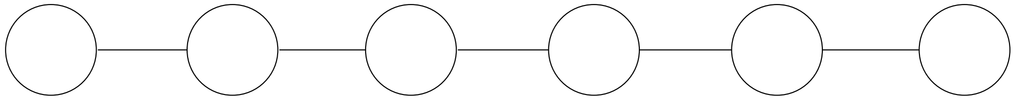


FIGURE 6. A chain with 6 beads

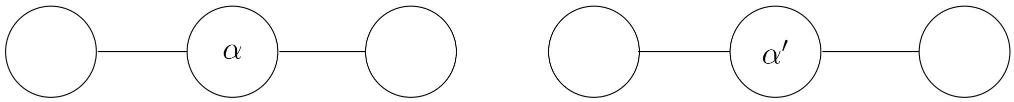


FIGURE 7. The 2 identical chains, α and α'

Using the Mirror Lemma 4.3 above, we know that any two identical chains are Type P. So the original position is Type N because the winning strategy is for the next player to split the chain into two identical pieces.

Imagine Richard takes 1 bead from α. On Louise's turn, she can do the same on α.

FIGURE 8. α and α'

From this position, Louise can copy whatever position Richard moves to on α or α' with the opposite chain, which guarantees Louise the last move, therefore winning the game regardless of the other player's moves. ☐

Proposition 4.5. *Imagine a chain with 7 beads.*

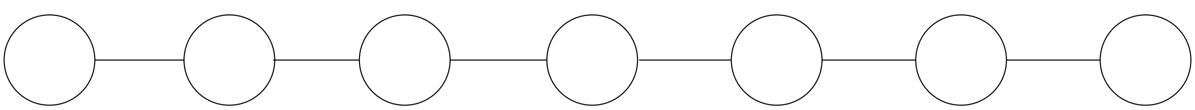
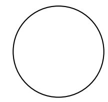


FIGURE 9. A chain with 7 beads

Again, Louise will go first. We know Louise will not choose to remove 1 bead because that will leave a 6-bead chain, which is Type N, giving Richard the win. If Louise removes 2 beads, the new 5-bead chain will be Type N. This is because Richard can remove 2 beads, leaving Louise at the Type P position in Example 4.1.

Before going into the proof for a winning strategy, we will go over some base cases of 1 through 4 beads, excluding 3 which was already done in 4.2

Proposition 4.6. *In this game with 1 bead, the next player to go will win, making this position Type N.*



Proposition 4.7. *In this case where there are 2 beads, the game is also Type N. The next player to move can take 2 beads in their turn.*

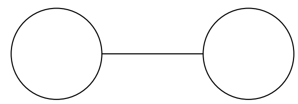
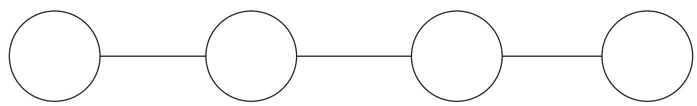


FIGURE 10. A chain with 2 beads.

Proposition 4.8. *In this case with 4 beads, the next player will remove 1 bead, so that the new position is Type P. This means that the previous position with 4 beads is Type N.*



Theorem 4.1. *If* $n \equiv 3 \pmod 4$*, then the* n *bead game is Type P. Otherwise, when* $n \not\equiv 3 \pmod 4$*, the game is Type N.*

Proof. Assume $n \equiv 3 \pmod 4$ is Type P and that the conjecture is true for all $k < n$. Since n is odd, our game tree will have branches from the root node representing the removal of 1 or 2 beads.

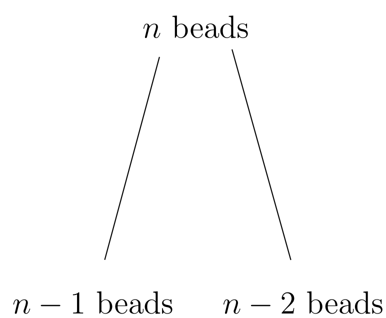


FIGURE 11. Game tree for an n bead game where $n \equiv 3 \pmod 4$

Since $n-1$ and $n-2 \not\equiv 3 \pmod 4$, we have by assumption that the $n-1$ and $n-2$ bead positions are Type N. It follows that our n-bead game is Type P. Inductive Step: Assume $n \not\equiv 3 \pmod 4$ and the conjecture is true for $k < n$.

If $n \equiv 0 \pmod 4$ or $n \equiv 2 \pmod 4$, then n is even, so Player 1 can cut the beads in half and win as demonstrated in Example 4.4. This follows that the n bead game is Type N in the case that n is even.

Finally, suppose that $n \equiv 1 \pmod 4$ is odd.

The winning strategy in this case is to take away 2 beads because $n-2 \equiv 3 \pmod 4$. By inductive assumption, the $n-2$ position is Type P, so the n bead game is Type N. ☐

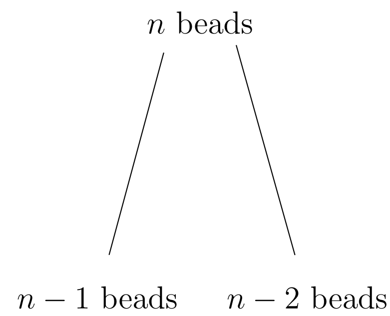


FIGURE 12. Game tree for an n bead game where $n \equiv 1 \pmod 4$

5. Acknowledgements

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References

[1] M. DeVos and D. A. Kent. *Game Theory: A Playful Introduction*. *American Mathematical Society* Vol. 80, 2016.