Combinatorial Hikita Conjecture

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Permutations

We call a permutation an ordered set of numbers $\{1, 2, \dots, n\}$ without repetitions. Number n in this case is called the permutation size. We will mostly refer to permutations as a set of bijections from a set S to itself. All permutations of the set $S = [N] = \{1, \dots, N\}$ form the (symmetric) group $Sym(n)$ eqipped with the operation of function composition.

One way to record a permutation is in the form of a table:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ a_1 & a_2 & a_3 & \dots & a_n \end{pmatrix},$$

which means that π places number a_i on i-th position.

For example, $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$ corresponds to an ordered set $\{3, 4, 2, 1\}$.

Permutations

Every permutation can be expressed as a product of a simple transpositions, that is transpositions $t = (i, i+1)$ that swaps i-th and $(i+1)$-th numbers. The minimal number of them needed to express a permutation σ is the length of σ.

For example, $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} = (2,3) \circ (1,2) \circ (3,4) \circ (2,3) \circ (1,2)$

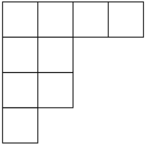
Young tableaux

A partition of a number N is any set of integers $\lambda_1 \geq \cdots \geq \lambda_k$ such that $\lambda_1 + \cdots + \lambda_k = N$. For example, $9 = 4 + 2 + 2 + 1$.

Young diagram of a partition $N = \lambda_1 + \cdots + \lambda_k$ is a set of rows justified to the left side, such that the top one has exactly λ_1 cells, the second λ_2, ..., the last one has λ_k.

We put numbers into the cells of the Young diagram. If numbers in all rows and columns are increasing, it is **Young tableau**. We denote the set of all Young tableaux that have n cells as $YTableaux(n)$ and those that have shape λ as $YTableaux(\lambda)$.

1	2	6	8
3	7		
4			
5			



Motivation for RS-algorithm

It is well-known, that for any group G, $|G| = \sum_{f \text{ is irrep}} (\dim f)^2$. In particular, it is true for $G = Sym(n)$. The irreps of a symmetric group are Young diagrams, and their dimensions are enumerated with Young tableaux. In this case, Robinson-Schensted algorithm gives simple combinatorial interpretation of the identity

$$n! = \sum_{\lambda \in \mathbb{Y}_n} |YTableaux(\lambda)|^2.$$

Robinson-Schensted algorithm gives us a bijection:

$$Sym(n) \longleftrightarrow \bigcup_{\lambda} YTableaux(\lambda) \times YTableaux(\lambda)$$

i.e. there is a bijection between the permutations of length n and pairs of Young tableaux that have n cells.

RS-algorithm: Insertion procedure

Insertion is usually denoted as $T \leftarrow x$, where T is a tableau and x is a value we insert. The row bumping algorithm looks the following way:

1. Keep a coordinate pair (i, j), initially set to $(1, k + 1)$ where k is the first row's of T length.
2. Find the first square in i-th row with an entry larger than x (or no such an entry), for example, by running a cycle 'while $j > 1$ and $x < T_{i,j-1}$'.
3. If (i, j) is empty, add it with x. Otherwise, swap x and $T_{i,j}$, go to the next row (increase i by one) and return to second step.

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 7 & 9 \\ \hline 3 & 8 \\ \cline{1-2} 4 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array} \leftarrow 5 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & \mathbf{5} & 9 \\ \hline 3 & \mathbf{7} \\ \cline{1-2} 4 & \mathbf{8} \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$$

RS-algorithm: Scheme

For $\pi \in S_n$ the algorithm is

1. P_0, Q_0 are empty tableaux.
2. $P_i = P_{i-1} \leftarrow \pi_i$ (by row bumping); add a new cell of P_i with entry i to Q_i.
3. Return (P_n, Q_n).

W. Fulton, Young Tableaux: With Applications to Representation Theory and Geometry

For example if $w = 45132$ we have:

$$P^{(1)} = \begin{array}{|c|}\hline 4 \\ \hline \end{array} \quad P^{(2)} = \begin{array}{|c|c|}\hline 4 & 5 \\ \hline \end{array} \quad P^{(3)} = \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 4 \\ \cline{1-1} \end{array} \quad P^{(4)} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 & 5 \\ \hline \end{array} \quad P^{(5)} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1} \end{array}$$

$$Q^{(1)} = \begin{array}{|c|}\hline 1 \\ \hline \end{array} \quad Q^{(2)} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \quad Q^{(3)} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \quad Q^{(4)} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad Q^{(5)} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1} \end{array}$$

Hence $w = 45132 \sim \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1} \end{array} \right)$.

RS-algorithm: Inversion

Inversion theorem. If a permutation π corresponds to a pair (P, Q) than its inverse π^{-1} corresponds to the reversed-order pair (Q, P), that is

$$\pi \sim (P, Q) \iff \pi^{-1} \sim (Q, P).$$

We accent on two fruitful proofs of this theorem: Viennot's geometric construction and growth diagrams.

G. Viennot, Une forme geometrique de la correspondance de Robinson-Schensted

R. P. Stanley, Enumerative Combinatorics, Vol. 2

Vector v_λ

The block (for odd a) is

$$Bl(a) = \left[-\frac{a-1}{2}; \frac{a-1}{2}\right] \cap \mathbb{Z}$$

For example, $Bl(5) = \{-2, -1, 0, 1, 2\}$.

Let $N = \lambda_1 + \cdots + \lambda_n$ be a partition ($\lambda_1 \geq \cdots \geq \lambda_n$) such that all the summands have the same parity (here we assume odd). We construct an auxiliary vector

$$\tilde{v}_\lambda = (Bl(\lambda_1), Bl(\lambda_2), \ldots, Bl(\lambda_k))$$

Let v_λ be a sorted version of $\tilde{v}_\lambda$.

The longest element

As $Sym(n)$ acts on v_λ, we consider $Stab(v_\lambda) \subset Sym(n)$, which fixes the vector. Of course,

$$Stab(v_\lambda) = \prod_{x=-(\lambda_1-1)/2}^{(\lambda_1-1)/2} Sym(cnt_x),$$

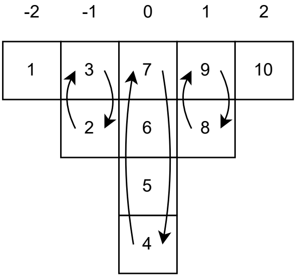
where cnt_x equals the number of occurences of x in v.

From each conjugancy class $\sigma Stab(v_\lambda)$ we take the longest element $w_0(\sigma)$ (one can prove the uniqueness). All the longest elements form the orbit:

$$Sym(n)/Stab(v_\lambda) \xrightarrow{w_0} Orb(v_\lambda).$$

If $\sigma = id$ we will omit it, so $w_0 = w_0(id)$.

The longest element: example



$$\lambda = (5,3,1,1),$$

$$\tilde{v}_\lambda = (-2,-1,0,1,2,-1,0,1,0,0),$$

$$v_\lambda = (-2,-1,-1,0,0,0,0,1,1,2),$$

$$w_0 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 3 & 2 & 7 & 6 & 5 & 4 & 9 & 8 & 10 \end{pmatrix}$$

$QOrb$-enumeration

Orbital elements that lie in the same left cell:

$$Orb(v_\lambda) \supset QOrb(v_\lambda) = \{w \in Orb(v_\lambda) \mid w \sim_L w_0\}.$$

As we work in $Sym(n)$ case, $w \sim_L w_0$ is equivalent to $Qw = Qw_0$.

Theorem ($QOrb$-enumeration)

There is a bijection between $QOrb(v_\lambda)$ and $YTableaux(\lambda)$.

Further reading: general

The overviews of the topic (the first one is elementary, concerns the symmetric case, the second one is more in-depth):

- G. Williamson, Mind your P and Q-symbols: Why the Kazhdan-Lusztig basis of the Hecke algebra of type A is cellular
- G. Lusztig, Hecke algebras with unequal parameters

Some conjectures can be found in the paper

- G. Lusztig, Some examples of square integrable representations of semisim-ple p-adic groups

Further reading: Robinson-Schensted

It turns out that there is a generalization of the Robinson-Schensted algorithm to some other groups.

J. Y. Shi, The generalized Robinson-Schensted algorithm on the affine Weyl group of type A_{n-1}

In the following article, Viennot's geometric construction was generalized in order to explain the previous paper combinatorially.

M. Chmutov, P. Pylyavskyy, E. Yudovina, Matrix-Ball Construction of affine Robinson-Schensted correspondence

Further reading: flags and algebra

This story goes much further into algebra. Still remaining in the 'combinatorial' world, we can notice the remarkable connections to flag varieties, refer to

- D. Rosso, Classic and Mirabolic Robinson–Schensted–Knuth Correspondence for Partial Flags
- Marc A. A. van Leeuwen, Flag Varieties and Interpretations of Young Tableau Algorithms

The general motivation for our problems is Springer correspondence

- Z. Yun, Lectures on Springer theories and orbital integrals
- J. P. Anker and B. Orsted, Lie Theory: Lie Algebras and Representations

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