Geometry of Minimal Surfaces with Layered Boundary Conditions

SPUR Final Paper, Summer 2015

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November 25, 2015

Abstract: It is a well known result that if a minimal surface is a graph over a convex region, then it is the unique surface spanning its boundary. In this paper we provide some basic structure results and counterexamples for more complicated boundary conditions.

1 Introduction

We will start with one theorem about the uniqueness of minimal surfaces.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be strictly convex and $\sigma \subset \mathbb{R}^3$ a simple closed curve which is a graph over $\partial\Omega$ with bounded slope. Then any minimal surface $\Sigma \subset \mathbb{R}^3$ with $\partial\Sigma = \sigma$ must be graphical over Σ and hence unique ([**CM**, p. 38]).*

We will study a more complicated case:

Let σ be a closed curve which can be projected onto a planar convex curve, Σ is a topological disk and also a minimal surface spanning σ. At each point on σ, there is exactly one tangent plane for Σ. σ has at most three layers and two turning points (see Fig. 1).

2 Background

We begin by giving the relevant background on minimal surfaces.

Theorem 2.1. *(Maximum principle) Let $\Omega \subset R^2$ be an open connected neighbourhood of the origin. If $u_1, u_2 : \Omega \to R$ are solutions of the minimal surface equation with $u_1 \leq u_2$ and $u_1(0) = u_2(0)$, then $u_1 \equiv u_2$ ([**CM**, p. 37-38]).*

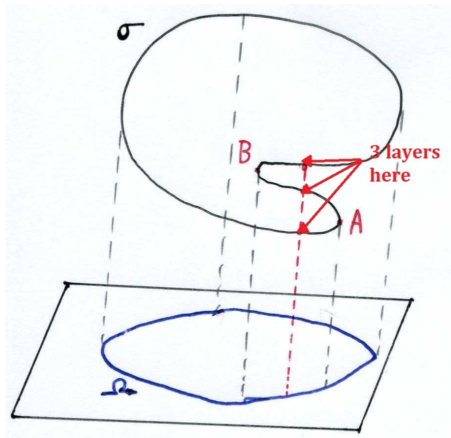

Figure 1: σ is the boundary and has 3 layers and 2 turning points

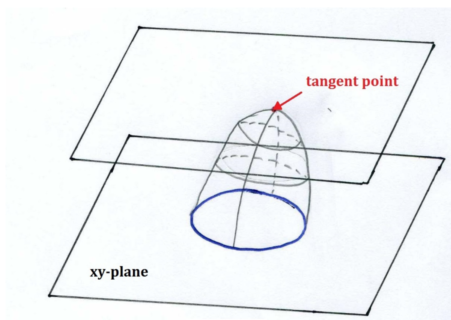

Figure 2: The plane tangent to Σ'

That is, if two minimal surfaces are tangent with each other and one of them is on one side of the other, then they are the same surface.

From this theorem, we get several corollaries easily.

Corollary 2.2. *If a minimal surface Σ' spans a planar curve σ', then it is a part of a plane.*

Proof: Assume that σ' is on the xy-plane. If Σ' is not contained in this plane, then we can find a point on Σ' whose z-component is not zero (let's assume that it's positive). Use a plane parallel to the xy-plane to approach the xy-plane from far away (where the z-component is large enough). When this plane first touch Σ', Σ' will be tangent to the plane and stay on one side of the plane (as shown in Fig. 2). Since the plane itself is a minimal surface, from the maximum principle we know that Σ' is just a part of a plane. □

Corollary 2.3. *If we can find a planar loop on Σ, then Σ is a part of a plane.*

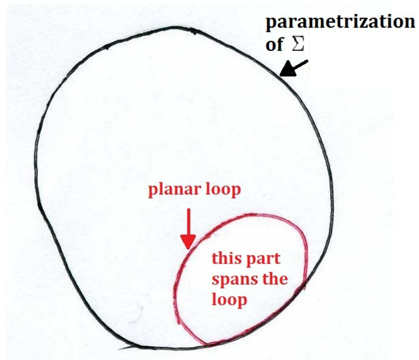

Figure 3: planar loop on Σ

Proof: As shown in Fig. 3, if there is a planar loop on Σ, then since Σ is disk-type, part of it is spanning this loop. According to Corollary 2.2, this part is contained by a plane. Then from the maximum principle we know that the whole surface is a part of a plane. □

Corollary 2.4. *If a loop σ can be projected onto the boundary of a convex region on a plane, then the minimal surface Σ spanning it must be contained in the cylinder with the planar convex region as its base (see Fig. 4).*

Proof: Assume that some points on Σ are outside of the cylinder. For example, assume that the plane is the xy-plane. There are some points on Σ whose y-component is larger than the y-component of any point in the planar convex region. Then we can use a plane parallel to the xz-plane to approach the cylinder from far away (where the y-component is large enough). When the plane first touches Σ, it is tangent to it and Σ lies on one side of the plane. From the maximum principle we know that Σ is just a part of a plane, which is a contradiction. □

With the same method, we can still prove:

Corollary 2.5. *If a closed curve lies on one side on a plane (part of it can be on that plane), then the minimal surface spanned by this curve lies in the same side of that plane.*

These are geometric properties for minimal surfaces. In fact, we also have an analytic one.

Proposition 2.6. *If Σ is a minimal surface, then at each point on Σ, we can find a local parametrization $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, which is both conformal and harmonic (here U is an open set in $\mathbb{R}^2$). Conversely, if $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ is a conformal and harmonic function, then the image of f is a minimal surface.*

This Proposition follows immediately after Section 2.6 in [**D**. p. 74-77].

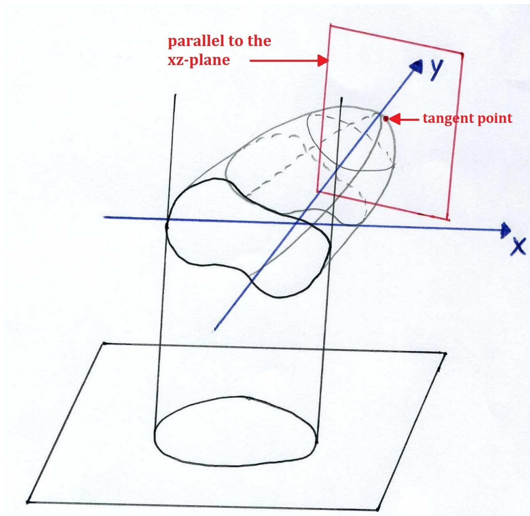

Figure 4: Σ lies out of the cylinder

3 Turning points

In order to study the layer structure of Σ, we can consider all the turning points.

Definition 3.1. *$P \in \Sigma$ is called a **turning point** if the normal vector at P is parallel to the projection plane (see Fig. 5).*

Proposition 3.2. *For any two points A and B on the projection of Σ, if we can find a path connecting them without touching the turning points, then the pre-images of A and B have the same number of points.*

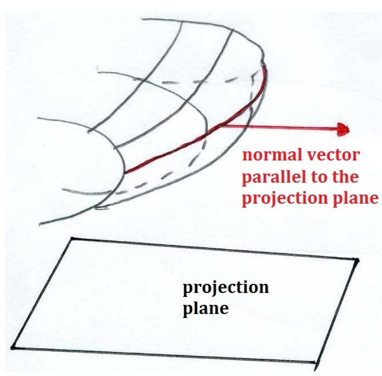

Figure 5: turning point

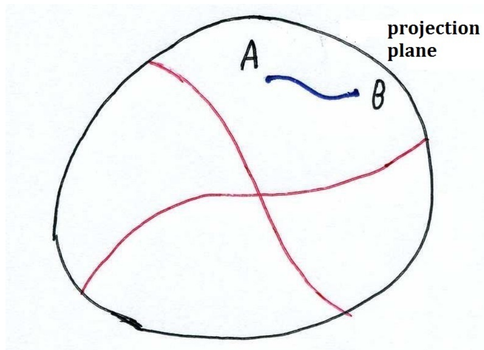

Figure 6: curves of turning points under projection

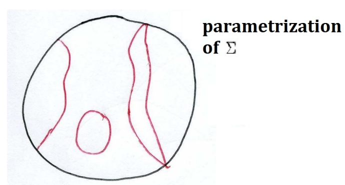

Figure 7: curves of turning points

Proof: For each point on Σ which can be projected onto A (let's call it A'), we can lift this path to be a path from A' to a point which can be projected onto B uniquely (let's call it B'). Conversely, this path can still be lifted from B' to A' uniquely. Therefore, the number of layers above A is the same as the one for B (see Fig. 6). □

According to this Proposition, if we can find out what these turning points look like (actually what the projection of them looks like), then we can find out the layer structure of Σ.

Proposition 3.3. *For each turning point on the interior of Σ, the Gauss map is a local diffeomorphism from Σ to S^2.*

Assume that the projection plane is just the xy-plane. Then the turning points are just the points which can be mapped to the equator of S^2 by Gauss map. If Gauss map is a local diffeomorphism, then locally the turning points have the same shape as the equator, and so form a smooth curve in $\mathbb{R}^3$. If we consider the disk which represents Σ (since Σ is disk-type), then the turning points are just several smooth curves (which can't intersect each other), as shown in Fig. 7.

Now let's prove proposition 3.3. We only need to prove that $|dN|$ is non-zero at any turning points (here N is the Gauss map) and then from the inverse

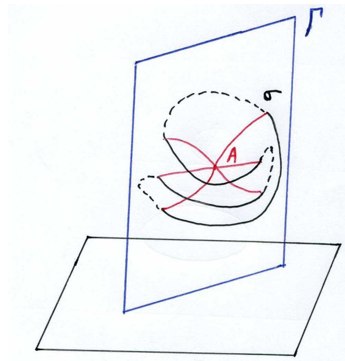

Figure 8: intersection between Γ and Σ

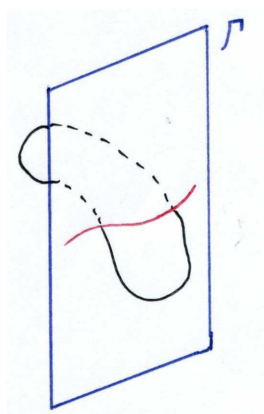

Figure 9: Σ near one curve of intersection

function theorem the Gauss map is a local differomorphism.

Lemma 3.4. *For each interior turning point A, if Γ is the tangent plane of Σ at A, then $\Gamma \cap \Sigma$ consists of several curves and it's impossible for six or more curves to go into A (see Fig. 8).*

Proof: Assume that there are at least six curves going into A.

First, we show that the intersection consists of several curves. According to corollary 2.3, there cannot be loops in $\Sigma \cap \Gamma$. The two parts of Σ (locally) separated by one curve lie on different sides of Γ, otherwise Σ would be tangent to Γ and would lie on one side of Γ near the tangent point. Then according to the maximum principle we would know that Σ is a part of a plane (see Fig. 9).

This also proves that for each point Q contained in the intersection of Σ and Γ, there are even number of curves going into it, since Σ goes to the different side of Γ when passing a curve, there must be even number of curves for Σ to

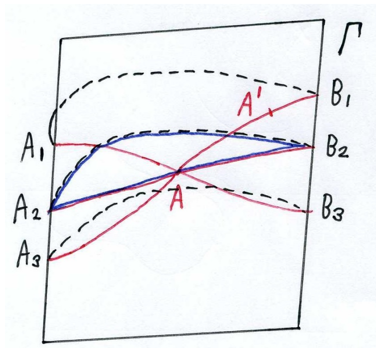

Figure 10: six curves of intersection going into A

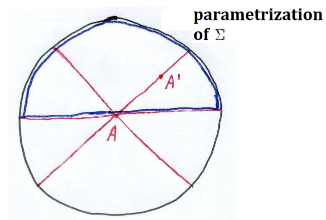

Figure 11: The pre-image of $\Sigma \cap \Gamma$

go back to the same side of Γ after going one circle around Q.

Since the boundary σ has at most six intersections with Γ (it's over the boundary of a convex region and has at most three layers) and there are no loops, there are at most six curves going into A (and each of these six branches goes to one end point on σ, as shown in Fig. 8). Therefore, the shape of the intersection is simple: six branches going into the same point A (see Fig. 8).

Since the boundary σ only has three layers and two turning-back points, on one side of Γ, there are three parts of σ connecting the six ending points of the intersection as shown in Fig. 10.

Consider the blue loop shown in Fig. 10. If we consider the disk representing Σ (see Fig. 11), part of Σ is spanned by that loop and some points (like A') in the intersection are contained in the interior of this part of Σ.

According to corollary 2.5, we know that that part of Σ spanned by the blue loop lies on one side of Γ. However, A' is also on Γ and it's not on the boundary of that part of Σ (which is just the blue loop). Therefore, the minimal surface spanned by the blue loop is tangent to Γ at A' and lies on one side of Γ, by the maximum principle we know that the whole surface is just a part of a plane, which is obviously impossible. □

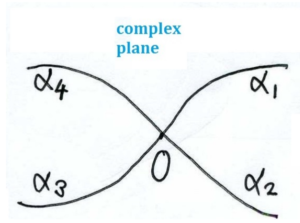

Figure 12: four curves near the origin

Proof of Proposition 3.3: Consider the local parametrization at point A. Assume that the Γ is the xy-plane. According to proposition 2.6, we can give a local conformal and harmonic parametrization $f(u,v) = (x(u,v), y(u,v), z(u,v))$ such that $f(0,0) = A$. Then as a component, $z(u,v)$ is also harmonic. Therefore, we can find another function $w(u,v)$, such that $z + iw$ is an analytic function about $\xi = u + iv$. Since the surface is not a plane, we can write $z + iw$ as $c_0 + c_1\xi^n + o(\xi^n)$, where n is a positive integer.

n is not 1 since Γ is the tangent plane, which implies $\partial z/\partial u = \partial z/\partial v = 0$.

As $u + iv$ goes once around the origin, $z + iw$ goes exactly n times around the origin. Then z will be positive and negative alternatively exactly n times. So there will be exactly $2n$ curves going into A in the intersection of Σ and Γ. Since we have proved that there are at most 4 such curves, n is exactly 2.

Therefore, the second derivatives of f cannot be all zero, so dN, as a matrix, is non-zero. Thus $|dN| \neq 0$. □

From the above discussion of $z + iw$, we know that:

Corollary 3.5. *There are exactly four curves going into the interior turning point A in the intersection of Σ and Γ and the angle between adjacent curves is exactly $\pi/2$.*

Proof: Since $z + iw = c_0 + c_1\xi^2 + o(\xi^2)$, if we consider the complex plane for ξ, then the set of ξ for which $z = 0$ is four curves near the origin (see Fig. 12, let's call them α_1, α_2, α_3, α_4) and the angle between each adjacent two curves is $\pi/2$. Since the parametrization f is conformal, so the right angle is preserved. Therefore, $f(\alpha_i)$, $i = 1, 2, 3, 4$, have the same local shape, which means, the angle between the adjacent two curves is exactly $\pi/2$. □

Since we have assumed that there are exactly two turning points on the boundary σ, from proposition 3.3 we know that the shape of all turning points should be smooth loops in the interior of Σ and curves which starts and ends at the two turning points on σ (see Fig. 13). Now let's consider the two turning points A and B on the boundary σ.

Proposition 3.6. *At each turning point (let's call it point A) on the boundary σ, there is exactly one curve of turning points going into A.*

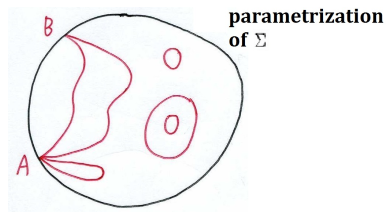

Figure 13: shape of turning points

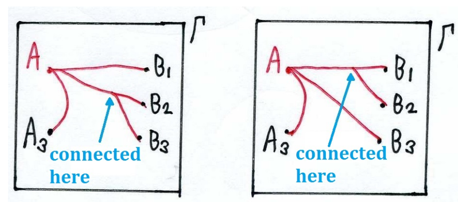

Figure 14: intersection of Γ and Σ

From this proposition, since there are exactly two turning points on the boundary σ, there must be a curve connecting them.

We can use a similar method as in the proof of Lemma 3.4. Consider the tangent plane Γ of Σ at point A. Now the intersection of Σ and Γ has at most 5 end points (see Fig. 14, two of them are overlapped at A).

Lemma 3.7. *In the intersection of* Σ *and* Γ*, there are at most 2 curves going into point* A*.*

Proof: Assume that there are three curves going into A, since there are at most 4 other end points, three of them are connected to A, then the rest one should also be connected (see Fig. 14). Now we can still find a loop (see the blue loop in Fig. 15) such that a part of Σ spans it and this part will contain one of A' and A'' in its interior. Then we can use the maximum principle again

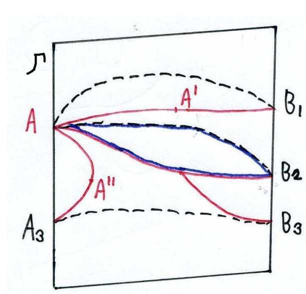

Figure 15: three curves going into A

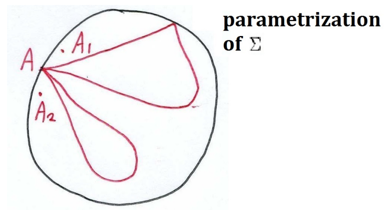

Figure 16: A_1 and A_2 on Σ

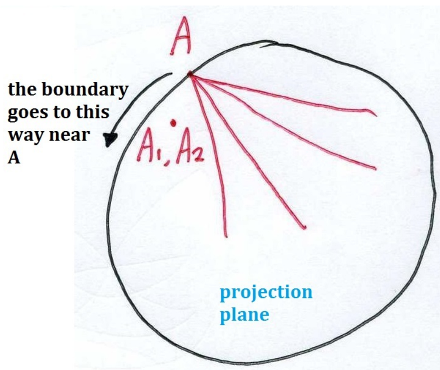

Figure 17: A_1 and A_2 on the projection

to get the contradiction.

There are three other possible shapes for the intersection of Σ and Γ, but the proof is similar for each of them. □

Lemma 3.8. *There are odd number of curves of turning points going into A.*

From this Lemma, if there are more than one curve of turning points going into A, then there will be at least three.

Proof of Lemma 3.8: Assume that there are even number of curves of turning points going into A, as shown in Fig. 16, which is the disk representing the surface. Since the Gauss map is a local diffeomorphism at interior turning points, when passing each curve in the disk, the Gauss map will point to different hemispheres. Since there are even number of curves of turning points going into A, for each pair of points A_1, A_2 (see Fig. 16) which is close enough to A, the Gauss map on them must point to the same hemisphere.

Now we choose the points A_1, A_2 in the following way. As shown in Fig. 17, which is the projection of Σ. We choose A_1, A_2 which can be projected onto the same point. Now consider the straight line l passing A_1 and A_2. We can choose A_1, A_2 close enough to the boundary so that l will not be tangent to Σ. The topology between l and σ is clear: l goes through the loop σ once. Since Σ is disk-type, so for each successive two intersections between Σ and l, l must

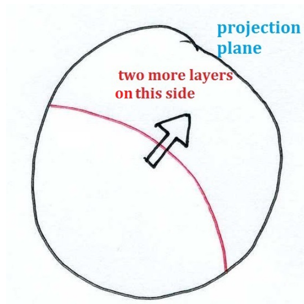

Figure 18: gain two more layers when passing a curve

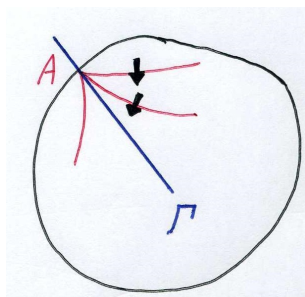

Figure 19: $\Gamma \cap \Sigma$ in projection direction

go through Σ from different sides. However, for points A_1 and A_2, they are successive intersections between l and Σ because when they are close enough to the boundary, the surface will have exactly three layers. Since the Gauss map points to the same hemisphere at A_1 and A_2, l go through Σ from the same side at these two points, which is a contradiction. □

Proof of Proposition 3.6: (See Fig. 18) Consider the projection of Σ. When we pass the smooth part of a curve of turning points in the direction shown in Fig. 18, there will be two more layers. Now consider the curves of turning points going into A (see Fig. 19), assume that there is more than one curve. Then according to Lemma 3.8, there are at least three curves. Since all these curves go into A from the direction of Γ, at least two of them are on the same side of Γ. There are two more layers when passing the curve in the direction shown in Fig. 19. Since these two curves go into A, so there should be at least four layers going into A. Each layer will intersect with Γ at one curve. Then there will be at least four curves of intersection on Γ, which contradicts with Lemma 3.7. □

Now the shape of turning points is more clear. There are several smooth loops and one smooth curve between the two turning points on the boundary (see Fig. 20). We will discuss the loops and that smooth curve in the following

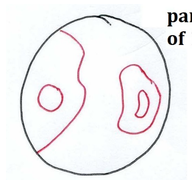

Figure 20: shape of turning points

two sections.

4 Loops of turning points

We can give σ some restrictions to eliminate all loops of turning points.

Proposition 4.1. *Let z have exactly 1 local maximum and minimum on σ, then there will be no loops of turning points on Σ.*

Proof of Proposition 4.1: Here we assume that the projection plane of the surface is the xy-plane. Assume that there is a loop of turning points on Σ (let's call this loop α). Use Σ' denote the part of surface spanned by α (including the boundary α).

Consider the Gauss map N. N is a continuous function from Σ' to the unit sphere S^2 and maps α to the equator. Then at least one of the poles of S^2 will be contained in $N(\Sigma')$. Otherwise since α is contractible in Σ', so $N(\alpha)$ should also be contractible in $N(\Sigma')$, hence also in S^2 with two poles removed. Since at each point on α, N is a local diffeomorphism, the image of α under N should go along the equator in one direction for several rounds. Then $N(\alpha)$ is not contractible in S^2 with two poles removed, which is a contradiction.

Therefore, we can find a point P on Σ', such that normal vector at P is parallel to the z-axis, which means that the tangent plane Γ of Σ at P is parallel to the projection plane.

Now let's consider the intersection of Σ and Γ, which consists of several curves. According to the restriction here for σ, there will be at most two end points. Also, according to corollary 2.3, there are no loops in the intersection. As a result, in $\Sigma \cap \Gamma$, there are exactly two curves going into point P (see Fig. 21).

According to proposition 2.6, we can give Σ a local harmonic conformal parametrization at P, let's call it $f(u,v) = (x(u,v), y(u,v), z(u,v))$. Then $z(u,v)$ is also harmonic, so we can find another function $w(u,v)$ such that $z+iw$ is analytic for $\xi = u + iv$. We can write $z + iw$ as $c_0 + c_1\xi^n + o(\xi^n)$ ($n \in \mathbb{Z}^+$). $n \neq 1$ since the normal vector at P is parallel to the z-axis, so $n \geq 2$. As we have shown in the proof of proposition 3.3, there are exactly $2n$ curves (at least 4 curves) going into P, which is a contradiction. □

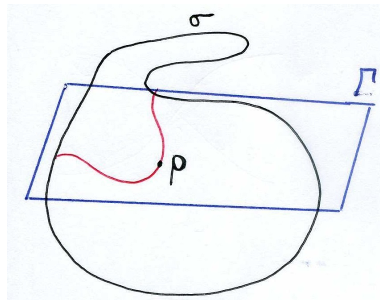

Figure 21: two curves going into P

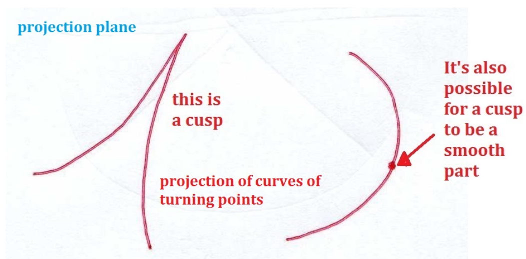

Figure 22: shape of cusps in projection plane

5 Cusps and multiple layers

In this section we will discuss the curve of turning points connecting the two turning points on σ.

At each interior turning point, the Gauss map is a local diffeomorphism, so we can give a parametrization $\alpha(t) = (x(t), y(t), z(t))$ to this curve of turning points. Then the tangent vector is $\alpha'(t) = (x'(t), y'(t), z'(t))$. Now consider the projection of α, which we call α_0. The tangent vector of it is $\alpha_0'(t) = (x'(t), y'(t))$. Therefore, $\alpha_0(t)$ is still smooth if $(x'(t), y'(t)) \neq (0, 0)$. If $(x'(t), y'(t)) = (0, 0)$, we call $\alpha_0(t)$ a **cusp** at this point.

As shown in Fig. 22, at each cusp, α_0 may make a rapid turning. Also, as shown in Fig. 16, when passing α_0 from one direction, then we can get two more layers. As a result, if we have multiple cusps along α_0, then it may be possible to get multiple layers for Σ.

Proposition 5.1. *For an arbitrarily large positive integer N, it's possible for Σ to have at least N layers at some points (which means the projection can map N points to the same point).*

We will just give the example for the case N=5. For larger N, the method to construct Σ is similar.

Example for the 5-layer surface:

According to proposition 2.6, we can give the example of Σ by giving the conformal harmonic parametrization $f(u,v) = (x(u,v), y(u,v), z(u,v))$. In order to make F harmonic, we only need to satisfy that the functions $f_1 = x_u + ix_v$, $f_2 = y_u + iy_v$, $f_3 = z_u + iz_v$ are analytic about $\xi = u + iv$. Then f is conformal $\Longleftrightarrow |f_u| = |f_v|$ and $\langle f_u, f_v \rangle = 0 \Longleftrightarrow f_3^2 = -f_1^2 - f_2^2$.

First, we will give the expression of f_1, f_2, f_3. Let

$$f_1(\xi) = \xi(\xi - 1)(\xi + 1) + i(T + \xi) \tag{1}$$

$$f_2(\xi) = (\xi + S)f_1(\xi) \tag{2}$$

Now $f_3^2 = -f_1^2 - f_2^2$ becomes $f_3^2 = -((\xi + S)^2 + 1)f_1^2$, so we can let

$$f_3 = if_1 g \tag{3}$$

Here g is an analytic function satisfying $g^2 = (\xi + S)^2 + 1$, which can be defined when $|\xi|$ is much smaller than S.

We define these three functions in the region $\{-100 \leq u \leq 100, -1 \leq v \leq 1\}$.

Proposition 5.2. $\{f(u,0), u \in \mathbb{R}\}$ *is the set of all the turning points.*

Proof: $f(u,v)$ is a turning point $\Longleftrightarrow N$ (normal vector) is perpendicular to the z-axis $\Longleftrightarrow x_u y_v = x_v y_u$ (by calculating N) $\Longleftrightarrow f_2/f_1 \in \mathbb{R}$ (since f_1 cannot be 0 when $|\xi|$ is small) $\Longleftrightarrow \xi \in \mathbb{R}$. □

Proposition 5.3. $f(1,0), f(0,0), f(-1,0)$ *are the only three cusps.*

Proof: Since the curve of turning points α is the image of the u-axis, so we can write $\alpha(t)$ as $f(t,0)$, then $\alpha'(t) = f_u$ and $\alpha_0'(t) = (x_u, y_u)$.

Therefore, $\alpha_0'(t)$ is a cusp $\Longrightarrow x_u = 0 \Longrightarrow \xi = 0, 1$ or -1. Conversely, When $\xi = 0, 1$ or -1, then $x_u(\xi) = y_u(\xi) = 0$, so α' reaches a cusp. □

Now we can get the expression of the surface in the form of some integrals:

$$f(u_0, v_0) = \int_0^{u_0} f_u(u,0)du + \int_0^{v_0} f_v(u_0, v)dv \tag{4}$$

The curve α is just $f(t,0)$. As shown in Fig. 23, the red curve is α_0. Now we only need to find a neighbourhood U on uv-plane containing the segment from -1 to 1 (and the boundary of U intersects with the u-axis at exactly two points), satisfying that:

(1) The image of $f(U)$ under the projection is a convex region and the boundary of $f(U)$ is mapped to the boundary of that convex region (call it β);

(2) $f(U)$ is embedded.

Then $f(U)$ will be the surface whose boundary has 3 layers and 2 turning points and can be projected onto a convex curve.

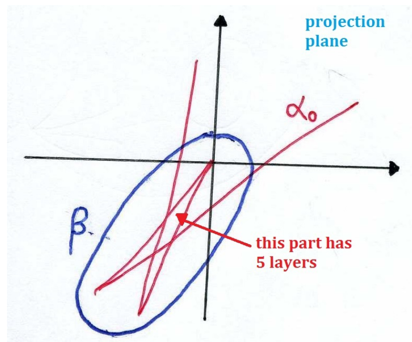

Figure 23: shape of the 5-layer surface

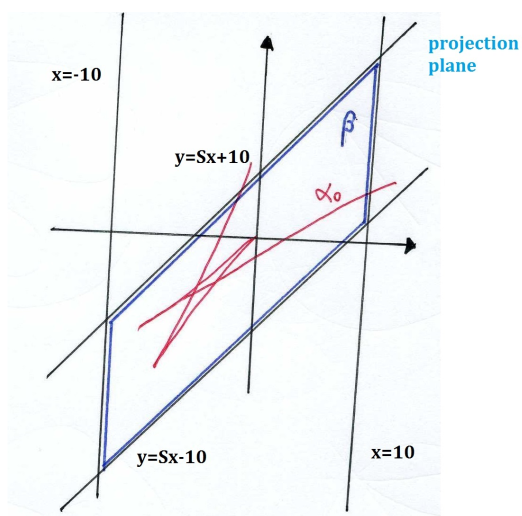

Figure 24: choice of β

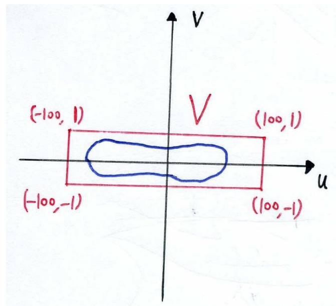

Figure 25: V and the neighbourhood inside V

The number of the layers of $f(U)$ is shown in Fig. 23, where $f(U)$ has 5-layers at some points (the method of counting the layers is shown in Fig. 18). Now we only need to find such U.

First, by direct calculation, we have

$$x(u,v) = \frac{1}{4}u^4 - \frac{1}{2}u^2 + \frac{3}{2}u^2v^2 - \frac{1}{4}v^4 - \frac{1}{2}v^2 + Tv + uv \tag{5}$$

$$y(u,v) = Sx(u,v) + \frac{1}{5}u^5 - \frac{1}{3}u^3 - uv^4 - uv^2 + Tuv + u^2v + 2u^3v^2 - \frac{1}{3}v^3 \tag{6}$$

Therefore, as shown in Fig. 24, the part of the curve α_0 from -2 to 2 is between the parallel lines $y = Sx + 10$ and $y = Sx - 10$, and also between $x = -10$ and $x = 10$. We choose β to be the parallelogram formed by these four lines.

Consider the rectangle V on the uv-plane shown in Fig. 25. If the image of $f(\partial V)$ under the projection is completely out of β ,then we can find a pre-image for β inside V on the uv-plane, which means we can find a neighbourhood U inside V whose boundary is mapped onto β.

In fact, we can prove that the projection of $f(V)$ is out of β by some simple calculations. If $|v| = 1$, then $|x(u,v)|$ will be very large since T is large enough, so it will not stay between $x = -10$ and $x = 10$. If $|u| = 100, |v| \leq 1$ and $x(u,v) = p \in [-10, 10]$, then we have

$$Tv = p - \frac{1}{4}u^4 + \frac{1}{2}u^2 - \frac{3}{2}u^2v^2 + \frac{1}{4}v^4 + \frac{1}{2}v^2 - uv \tag{7}$$

so

$$y - Sx = \frac{1}{5}u^5 - \frac{1}{3}u^3 - uv^4 - uv^2 + Tuv + u^2v + 2u^3v^2 - \frac{1}{3}v^3 = -\frac{1}{20}u^5 + h(u,v) \tag{8}$$

here $|h(u,v)|$ is much smaller than $|\frac{1}{20}u^5|$ since $|u| = 100$ and $|v| \leq 1$. Therefore, $|y - Sx|$ will be very large so it's not between $y = Sx + 10$ and $y = Sx - 10$.

Now we only need to prove that $f(U)$ is embedded. We only need to prove that $f(V)$ is embedded. Assume that we can find two points $(u_1, v_1), (u_2, v_2)$,

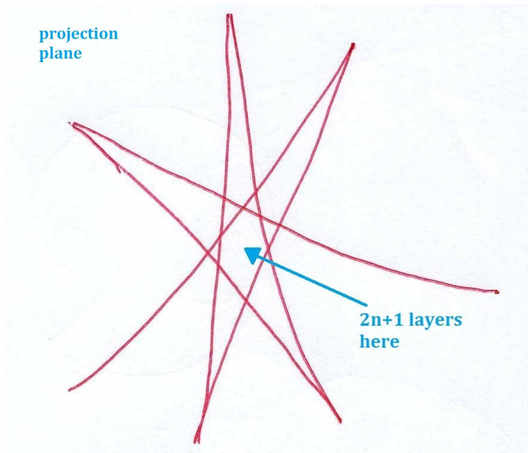

Figure 26: shape of surface with multiple layers

such that $f(u_1, v_1) = f(u_2, x_2)$. Then

$$0 = f(u_2, x_2) - f(u_1, v_1) = \int_{u_1}^{u_2} f_u(u, v_1)du + \int_{v_1}^{v_2} f_v(u_2, v)dv \quad (9)$$

Consider the x-component:

$$\int_{u_1}^{u_2} x_u(u, v_1)du + \int_{v_1}^{v_2} x_v(u_2, v)dv = 0 \quad (10)$$

Since there is a T in x_v, so $|x_v| >> |x_u|$ inside V. Therefore we have $|u_2 - u_1| >> |v_2 - v_1|$.

However, we can also consider the z-component:

$$\int_{u_1}^{u_2} z_u(u, v_1)du + \int_{v_1}^{v_2} z_v(u_2, v)dv = 0 \quad (11)$$

We have $f_3 = igf_1$. $g^2 = (\xi + S)^2 + 1$, so g will be very close to S since S is large enough. Therefore, $|x_v| >> |x_u| \Longrightarrow |z_u| >> |z_v|$. Now we get the opposite result $|u_1 - u_2| << |v_1 - v_2|$, which is a contradiction.

Now we get an example for a surface whose boundary has only three layers and two turning points to have five layers at some points. In fact, we can use the same method to construct a surface with even more layers. We just need to choose

$$f_1(\xi) = \xi(\xi - 1)(\xi - 2)(\xi - 3).....(\xi - 2n + 2) + i(\xi + T) \quad (12)$$

Then the shape of α_0 is shown in Fig. 26, where we have $2n + 1$ layers at some points (we can still use the previous calculations).

Now we finish the proof of proposition 5.1. □

6 Area bounds for multiple layers

We can give an upper bound for the area of the surface where there are multiple layers.

Proposition 6.1. *Let Σ be absolutely minimizing for its boundary conditions. If the projection of Σ is a disk of radius R and the height of Σ is $h < R$, then on the projection of Σ, the area where Σ has at least n layers is no more than*

$$\frac{\pi R^2 + 2\pi Rh + 4\pi h^2}{3(n-1)} \tag{13}$$

We will use the following theorem:

Theorem 6.2. *If Σ is an absolutely minimizing minimal surface, and B is a ball with radius r, then*

$$Area(B \cap \Sigma) \leq \frac{4}{3}\pi r^2 \tag{14}$$

*([**CM**, p. 77])*

Proof of Proposition 6.1: Since the projection of Σ is a disk of radius R and the height of Σ is h, we can find a ball B with radius $R+h$ to contain Σ. Then $Area(B \cap \Sigma) = Area(\Sigma)$. $Area(\Sigma)$ is no less than the area of its projection. If we use A to denote the area where Σ has at least n layers, then the area of the projection is at least $\pi R^2 + (n-1)A$. According to Theorem 6.2, we have

$$\pi R^2 + (n-1)A \leq \frac{4}{3}\pi(R+h)^2 \tag{15}$$

So we get

$$A \leq \frac{\pi R^2 + 2\pi Rh + 4\pi h^2}{3(n-1)} \tag{16}$$

□

7 Future work

We also have some future works on this problem. Although we have some counter examples, but all the surfaces we get in section 5 must be quite tall, since we need T and S to be large enough. So if we assume that the surface is relatively flat compared to its projection (or some other reasonable restriction), then maybe it's necessary for the surface to have only three layers in its interior.

8 Acknowledgement

I wish to thank Prof. Jerison and Prof. Moitra for their useful suggestions. I specially thank my mentor Paul Gallagher for all the time he has spent and the invaluable help he has given me. Finally I'm also grateful to the staff of MIT SPUR 2015 for producing this excellent opportunity for me to experience math research.

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