ON THE COHEN-MACAULAYNESS OF CERTAIN HYPERPLANE ARRANGEMENTS

SPUR FINAL PAPER, SUMMER 2014

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Abstract. Inspired by the representation theory of rational Cherednik algebras, we consider whether certain rings corresponding to hyperplane arrangements in $\mathbb{C}^n$ satisfy the Cohen-Macaulay property. Specifically, for a partition $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$ we define a certain variety $X_\lambda \subset \mathbb{C}^n$ and study its coordinate ring $\mathbb{C}[X_\lambda]$. The question which we work towards is: for which λ is X_λ Cohen-Macaulay? We consider the sub-question of when the coordinate ring of X_λ/S_n, defined below, is Cohen-Macaulay, and compute related information such as its Hilbert series and formulas for generating higher degree polynomials in terms of lower-degree generators.

1. Preliminaries

1.1. **Dimension.** Let us briefly define an affine algebraic variety, and its corresponding ring of functions. These rings of functions, for a specific class of varieties, will be the main object of our study.

Let $\mathbb{C}[x_1, \dots, x_n]$ denote the ring of polynomials with coefficients in $\mathbb{C}$, and let $\mathbb{V}(\{f_i\}_{i\in I})$ denote the intersection of zeros of the polynomials $\{f_i\}$, i.e. $\mathbb{V}(\{f_i\}) = \bigcap_{i\in I} f_i^{-1}(0)$. As a convention, we usual write $\mathbb{A}^n$ instead of $\mathbb{C}^n$ when we are talking about the domain of a polynomial or set of polynomials, so $\mathbb{V}(\{f_i\}) \subset \mathbb{A}^n$.

Definition 1. A *complex affine algebraic variety* (which we will normally refer to as just an *affine variety*) is a subset $V = \mathbb{V}(\{f_i\}) \subset \mathbb{A}^n$ for some set of polynomials $\{f_i\} \subset \mathbb{C}[x_1, \dots, x_n]$.

The collection of affine varieties $V \subset \mathbb{A}^n$ forms a topology of closed sets, called the *Zariski topology*.

A *subvariety* of V is a closed subset of V in the subspace topology. A variety V is *irreducible* if it cannot be written as the nontrivial union of two closed subsets.

Date: July 30, 2014.

Clearly $\mathbb{V}(\{f_i\}) = \mathbb{V}(\langle f_i \rangle)$ since finite sums of polynomials that vanish still vanish, i.e. we may as well assume that the collection of $\{f_i\}$ forms an ideal in $\mathbb{C}[x_1, \ldots, x_n]$. We naturally get a partially inverse construction – given an affine variety V, we can construct $\mathbb{I}(V) = \{f_i \in \mathbb{C}[x_1, \ldots, x_n] : f_i \text{ vanishes on } V\}$. Clearly this is again an ideal, and it is not hard to show that $\mathbb{V}(\mathbb{I}(V)) = V$ for any affine variety V. The content of Hilbert's Nullstellensatz is that these operations are inverses when we restrict to radical ideals, i.e. $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$.

If two polynomials f_1, f_2 are equal mod $\mathbb{I}(V)$ for some variety $V \subset \mathbb{A}^n$, then they restrict to the same function on V. This motivates the definition that the *coordinate ring* of V is the ring $\mathbb{C}[x_1, \ldots, x_n]/\mathbb{I}(V) \stackrel{\text{def}}{=} \mathbb{C}[V]$.

Using geometric intuition as a guide, we define the dimension of an algebraic variety as follows:

Definition 2. The *dimension* of an affine variety $V \subset \mathbb{A}^n$ is the largest d such that there is a strictly decreasing chain of irreducible subvarieties (i.e. nonempty intersections of varieties with V) $V = V_d \supsetneq V_{d-1} \supsetneq \ldots \supsetneq V_0$.

It is straightfoward to verify that a variety $V \subset \mathbb{A}^n$ is irreducible if and only if $\mathbb{I}(V) \subset \mathbb{C}[x_1, \ldots, x_n]$ is a prime ideal. Therefore we can define a corresponding algebraic definition of dimension, called the *Krull dimension*:

Definition 3. The *(Krull) dimension* of a ring R is the supremum of lengths of chains of prime ideals in R. If $I \subset R$ is an ideal then we write $\dim I$ to mean the dimension of R/I as a ring, i.e. $\dim R/I$.

Another definition is that of *codimension*:

Definition 4. For a prime ideal $I \subset R$, the *codimension of* I is the dimension of the localization R_I, localized at $R - I$.

It is not difficult to show this corresponds to our geometric intuition in the case that the ring R contains a subfield which it is finitely generated over (see: [Eis]). In general, this is true for irreducible varieties, but not in general, as the following example illustrates:

Example 1. Take the variety of a plane with a line intersecting it, and the corresponding coordinate ring $R = \mathbb{C}[x, y, z]/(xy, xz)$. We see that $\dim R = 2$, perhaps more easily by thinking of the ring R as the coordinate ring of a two-dimensional variety. Take the ideal $I = (x + 1, y, z) \subset R$, which corresponds to a point. Then $\dim I = 0$, yet $\operatorname{codim} I = 1$.

Here $\operatorname{codim} I = \dim R - \dim I$ is only true when we take R to be the coordinate ring of the irreducible component containing I. This is the sense in which the definition given is local – $\operatorname{codim} I$ only "sees" the dimension of the variety near the point corresponding to I.

1.2. **Depth.** The concept of depth is another algebraic concept that intuitively corresponds to the geometric notion of dimension. Depth essentially tells us how many equations we need to define our variety V. The Cohen-Macaulay condition on a ring R will then be a certificate that depth and dimension agree locally.

Definition 5. Let R be a ring and M a finitely generated R-module. A regular M-sequence is a sequence of elements $r_1, \ldots, r_n$ such that:

- The ideal $(r_1, \ldots, r_n)$ is proper, i.e. not equal to all of R
- The first element r_1 is a nonzerodivisor on M
- Inductively, r_{i+1} is a nonzerodivisor on $M/(r_1, \ldots, r_i)M$

The basic facts about regular sequences needed are proven using a homological tool known as the Koszul complex, which we define here:

Definition 6. Given a ring R, a free finitely generated R-module $N = R^n$, and an element $x \in N$, the corresponding *Koszul complex* is the complex

$$K(x) = 0 \to R \to N \to \Lambda^2 N \to \ldots \to \Lambda^n N \to 0$$

where each map is defined by $a \mapsto x \wedge a$ (here $\wedge$ denotes multiplication in the exterior algebra). The fact that $x \wedge x = 0$ exactly tells us that this is a complex.

Notice that $\Lambda^i N \cong R^{\binom{n}{i}}$, and in particular if $i > n$ then $\Lambda^i N = 0$. It is a straightforward argument in Eisenbud that $H^n(K(x)) = R/(x_1, \ldots, x_n)$, where $x = (x_1, \ldots, x_n)$ (here H^n is the homology at the last nonzero term in $K(x)$).

The fundamental result we need about the Koszul complex, which much of chapter 17 in Eisenbud is devoted to proving, is the following:

Theorem 1.2.1 (17.4 in Eisenbud)**.**

- *Let M be a finitely generated module over a ring R. Define $M \otimes K(x)$ as the complex $0 \to M \otimes R \to M \otimes N \to M \otimes \Lambda^2 N \to \ldots \to M \otimes \Lambda^n N \to 0$.*
- *If the first $r - 1$ homology groups of $M \otimes K(x)$ vanish, but $H^r(M \otimes K(x)) \neq 0$, then every maximal regular M-sequence contained in the ideal $I = (x_1, \ldots, x_n)$ has length r. Also, $K^n(M \otimes K(x)) = M/xM$, so there is such an $r \leq n$.*

Definition 7. The integer r guaranteed by the previous theorem is called the *depth* of I in M, written $\operatorname{depth}_R(I, M)$. When $M = R$ we write $\operatorname{depth} I$ for short.

A ring R is called *Cohen-Macaulay* if $\operatorname{depth}\mathfrak{p} = \operatorname{codim}\mathfrak{p}$ for all prime ideals $\mathfrak{p} \subset R$. Equivalently, R is CM iff this holds for all maximal ideals of R (see [Eis]).

Remark 1. We should note that in the case $(R, \mathfrak{m})$ is a local ring, this is equivalent to $\operatorname{depth}\mathfrak{m} = \operatorname{codim}\mathfrak{m} \overset{\text{def}}{=} \dim R_{\mathfrak{m}} = \dim R$. Moreover, it follows that a general ring R is CM iff every localization of it $R_{\mathfrak{m}}$, for $\mathfrak{m} \subset R$ a maximal ideal, is CM; that is, CM-ness is a local property.

Remark 2. This previous remark holds even if $(R, \mathfrak{m})$ is a *graded* local ring (to be defined below).

Example 2. Taking $R = \mathbb{C}[x, y, z]/(xy, xz)$ and $I = (x + 1, y, z)$ as before, we saw that $\operatorname{codim}_R I = 1$. However, we can see that $y \in I$ is a nonzerodivisor, and that $x+1 \in R/y = \mathbb{C}[x, z]/(xz)$ is a nonzerodivisor. Thus $(y, x + 1)$ form a regular sequence, thus a maximal regular-R sequence in I has length at least 2. Thus $\operatorname{depth}(I, R) \neq \operatorname{codim}_R I$, and thus R is *not* Cohen-Macaulay (hereby abbreviated CM).

This example illustrates the general fact that if a variety V has two irreducible components of different dimensions which intersect, then $\mathbb{C}[V]$ is not CM as a ring.

Example 3. Any curve (i.e. 1 dimensional affine variety V) is CM. An irreducible subvariety of a curve corresponds to a point, which has codimension 1, which is also the depth of the ideal which generates the point in the coordinate ring of the curve.

2. Previous results

Let $\lambda = (\lambda_1, ..., \lambda_r)$ be a partition of n. Then the equations

$$x_1 = \ldots = x_{\lambda_1}, x_{\lambda_1+1} = \ldots = x_{\lambda_2}, \ldots, x_{n-\lambda_r+1} = \ldots = x_n$$

define an r dimensional hyperplane, which we denote as E_λ. Letting S_n (the symmetric group on n letters) act on the x_i in the obvious way, we get $X_\lambda = S_n \cdot E_\lambda$, the union of some finite number of hyperplanes (at most $n!$ of them; it can be less, for example if $|\lambda| = 1$ then $E_\lambda = X_\lambda$). Thus X_λ is defined by the points for which *some* λ_1 coordinates are equal, some *other* λ_2 coordinates being equal, etc. Moreover, we see X_λ is still a variety, but it is not so obvious if it is CM or not – we have many different planes intersecting, but the dimension of each plane is r and thus X_λ is equidimensional (meaning all its irreducible components have the same dimension).

Let us note that we think of the coordinate rings of these varieties as *graded* rings, graded by degree of the polynomial.

Definition 8. A *graded ring* is a ring $R = \oplus_{i \geq 0} R_i$ such that $R_i \cdot R_j \subset R_{i+j}$.

A *homogeneous element* of a graded ring R is an element of some R_i. A *graded ideal* I of R is an ideal generated by homogeneous elements. A *graded local ring* is a ring with a unique maximal graded ideal.

All theorems we mention for local rings, such as the Auslander-Buchsbaum formula, have analogous versions for graded local rings. We don't call attention to this distinction much in the interest of avoiding repeating the word "graded" too much, but it is worth remembering throughout that R_λ is indeed a graded ring.

Obviously $(0, \dots, 0) \in X_\lambda$ for any λ, and if $x \in X_\lambda$ then so is cx for any $c \in \mathbb{C}$. Thus $X_\lambda = X_\lambda^0 \times \mathbb{C}$ where $X_\lambda^0 = \{x \in X_\lambda : \sum_{i=1}^r x_i = 0\}$. Also $\dim X_\lambda^0 = r - 1$, making X_λ^0 a more computationally amenable variety to consider (we describe the coding and computational aspects of this project later).

In this question we can replace X_λ with X_λ^0 (taking a product with an affine line $\mathbb{C}$ does not change the question). Also note that if X_λ is CM then X_λ / S_n is CM as well.

The following theorem was proved in [EGL], Proposition 3.11, and was motivated in the context of the paper by the representation theory of rational Cherednik algebras, which the author here is not at all qualified to discuss:

Theorem 2.0.2. *Suppose that $\lambda = (m, ..., m, 1, ..., 1)$. Then:*

(1) X_λ is CM if either the number of ones is $< m$, or $m \leq 2$;
(2) X_λ is not CM if $m \geq 3$ and the number of ones is at least m.

Example 4. We can see that if $\lambda = (p, q)$, i.e. $|\lambda| = 2$, then X_λ is a curve. By Example 3 above, we then see that X_λ must be CM.

A reduction of the conjecture can be obtained by using the technique of formal neighborhoods, again from Prop. 3.11 in [EGL]. Namely, arguing as in this proof, one can show:

Proposition 1. *Suppose $\ell \leq \lambda_r$. Then if $X_\lambda \subset \mathbb{C}^n$ is not CM, $X_{\lambda,\ell} \subset \mathbb{C}^{m+\ell}$ is not CM either.*

Proof. Consider the point $x = (1, \dots, 1, 0, \dots, 0) \in X_{\lambda,\ell}$ where the number of ones is ℓ. A formal neighborhood of x in $X_{\lambda,\ell}$ looks like the product of the formal neighborhood of zero in X_λ with a formal disk, which implies the statement. □

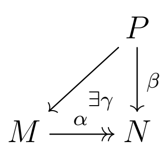
3. Computational tactics: theory

To consider the entire ring R_λ, which scales in dimension with $n = \lambda_1 + \ldots + \lambda_r$, is intractable for $n > 4$ with Macaulay. Theorem 2.0.2 above shows us for $\lambda = (p,p,p), (p,p,1)$, or $(p,1,1)$ where $p \geq 2$, that X_λ is CM. Computations previously done by Steven Sam in Macaulay2 also tell us that $(4,2,2)$ gives a CM ring, but we don't have a theoretical proof of this assertion. So, as a more computationally viable alternative, we study an associated ring of invariants. Let us describe now the various approaches we have used to glean information from the ring of invariants.

3.1. **Ring of invariants.** The ring of invariants, which we write as R, is the coordinate ring of X_λ/S_n, the variety where we identify points of X_λ in the same S_n-orbits. It can be shown if we write $\lambda = (\lambda_1, \ldots, \lambda_r)$ then this ring is generated by the polynomials of the form $P_i = \lambda_1 x_1^i + \ldots + \lambda_r x_r^i$; in fact by Hilbert's invariant theorem, Corollary 1.5 in [Eis], this ring is finitely generated. We can reduce the number of variables once more to $r-1$ by observing the following: as noted in the previous section, X_λ is CM iff X_λ^0 is CM; to consider the ring of invariants of X_λ^0 by projecting out one of the variables is the same as to impose $P_1 = 0$. Thus we can substitute $x_r = -\frac{\lambda_1}{\lambda_r}x_1 - \ldots \frac{\lambda_{r-1}}{\lambda_r}x_{r-1}$ in for all the other P_i, and thus consider instead $R = \mathbb{C}[X_\lambda^0/S_n]$.

3.1.1. *The Auslander-Buchsbaum formula and its applications.* Let us recall the notion of the projective dimension of a module:

Definition 9. A *projective module* is a module P such that for any epimorphism $\alpha : M \twoheadrightarrow N$ and map $\beta : P \to N$ there is a map $\gamma : P \to M$ such that $\alpha\gamma = \beta$, i.e. the following diagram commutes:



A *finite projective resolution* of an A-module M is an exact sequence of the form $K : 0 \to P_n \to P_{n-1} \to \ldots \to P_1 \to P_0 \to M \to 0$ where each P_i is projective. The *projective dimension* of M, written $\operatorname{pdim}_A M$, if it is finite, is the minimal such n that there exists a projective resolution of M of the previous form.

Fact 1. A free module is projective, and when $A = k[x_1, \ldots, x_n]$ the converse is true (this is the Quillen-Suslin theorem). This is much easier to prove in the graded case, which is albeit the one we are working with.

Also, every finitely generated module, i.e. our A, has finite projective dimension by Corollary 19.8, [Eis].

As projective dimension is a computationally amenable quantity, the following theorem is an important tool in detecting CM-ness:

Theorem 3.1.1 (Auslander-Buchsbaum formula)**.** *If $(A, \mathfrak{m})$ is a (graded) local ring, and M is a finitely generated (graded) A-module of finite projective dimension, then*

$$\operatorname{pdim}_A M = \operatorname{depth}_A(\mathfrak{m}, A) - \operatorname{depth}_A(\mathfrak{m}, M).$$

One important way we will use projective dimension is through the following. From now on, we set $A = \mathbb{C}[P_2, \ldots, P_r]$, and R is as in the previous section.

Theorem 3.1.2. *The ring R is CM iff $\operatorname{pdim}_A R = 0$.*

Proof. Let us note that $\operatorname{depth}_A A = \dim A$ since A is CM (it is a polynomial ring, hence corresponds to a nonsingular variety, i.e. affine space), and that $\dim A = \dim R$ since R is finite over A. Additionally, $\operatorname{depth}_A R = \operatorname{depth}_R R$.

As we noted before, since R is graded local, CM-ness is equivalent to $\operatorname{depth}_R R = \dim R$. By the previous remarks, we see then that this is equivalent to $\operatorname{depth}_A R = \dim R$ (the third remark), which is equivalent to $\operatorname{depth}_A R = \dim A$, which is equivalent to $\operatorname{depth}_A A - \operatorname{depth}_A R = 0$. But by the Theorem 3.1.1 this is equivalent to $\operatorname{pdim}_A R = 0$. ☐

Corollary 1. *The ring R is CM iff it is free over $A = \mathbb{C}[P_2, \ldots, P_r]$.*

Proof. Let us show that $\operatorname{pdim}_A R = 0$ is equivalent to R being free. Well, $\operatorname{pdim}_A R = 0$, by the definition of projective dimension, is equivalent to R being a projective A-module. It is then the content of the graded Quillen-Suslin theorem, mentioned earlier, that, for A a polynomial ring, R is free. ☐

The background on projective dimension might seem like overkill, since the previous theorem uses only the case $\operatorname{pdim}_A R = 0$ (which is only a lemma in the full proof of Auslander-Buchsbaum). However, the following additional theorem will be crucial in the `isCM` function:

Theorem 3.1.3. *If $(S, \mathfrak{m})$ is a CM graded local ring with a (graded) surjection $\varphi : S \to R$, and $I = \ker \varphi$, then R is CM iff $\operatorname{pdim}_S R = \operatorname{codim} I$.*

Proof. By Theorem 3.1.1 with $A = S$ we get that $\operatorname{pdim}_S R = \operatorname{depth}_S(\mathfrak{m}, S) - \operatorname{depth}_S(\mathfrak{m}, R)$ (we use φ to make R an S module in the obvious way:

$s \cdot r = \varphi(s)r$). Now since S is CM and graded local, $\dim S = \operatorname{depth} S$, and since φ is surjective we get $\operatorname{depth}_S(\mathfrak{m}, R) = \operatorname{depth} R$.

Now let $I = \ker \varphi$, so that $\operatorname{codim} I = \dim S - \dim R$, and $\operatorname{pdim}_S R = \dim S - \operatorname{depth} R$ by the above. Thus $\dim R = \operatorname{depth} R$ is equivalent to $\operatorname{codim} I = \operatorname{pdim}_S R$. □

4. Computational tactics

4.0.2. *The isCM method.* We can exploit Theorem 3.1.3 to determine in some cases whether or not R is CM. This is only computationally viable for simple cases of λ, such as where $|\lambda| \leq 3$ or $|\lambda| = 4$ and λ contains only two distinct elements. The method is not completely rigorous because it does not prove that the map $\varphi : S \to R$ is surjective, but rather makes sure that this is very likely, and its results have so far been consistent.

As we mentioned, the ring of invariants R is finitely generated by Hilbert's invariant theorem. We now describe the `guess(p,q,r)` method, which determines N such that $P_2, \cdots, P_N$ very likely generate R as a $\mathbb{C}$-algebra, in the case $\lambda = (p, q, r)$ (higher variable cases will be analogous).

The method `guess(p,q,r)` checks, for increasing N, whether P_{N+1} is in the subalgebra generated by $P_2, \cdots, P_N$. To check this, it calls `surf(p,q,r,N)`, which computes the relations between $P_2, \cdots, P_N$, for $N = N$ and $N = N + 1$. It then compares the Hilbert series of `surf(p,q,r,N)` and `surf(p,q,r,N+1)` and checks whether they are the same. It stops when this is the case, and this is the point at which our algorithm is not rigorous.

Note 1. For $\lambda = (1, 1, \ldots, 1)$, notice that if we don't set $P_1 = 0$ to eliminate a variable, R is the algebra of symmetric polynomials in r variables. Then the above method provably does work in this case, as a result of the fundamental theorem of symmetric polynomials, which in our notation gives us that $R = \langle P_1, P_2, \ldots, P_r \rangle$.

We now take $S = \mathbb{C}[v_2, \ldots, v_N]$, where $N =$`guess(p,q,r)`, and maps $v_i \mapsto P_i \in R$, which (empirically) gives us a surjection $\varphi : S \to R$. Then by Theorem 3.1.3, CM-ness of R is equivalent to $\operatorname{codim} I = \operatorname{pdim}_S R$. The method `isCM` precisely calculates both of these quantities.

4.0.3. *Hilbert series coefficients.* The previous works for simple cases, but for cases where $|\lambda| > 4$ or some where $|\lambda| = 4$, such as $(4, 3, 3, 1)$, does not terminate within a reasonable amount of time. A quicker method, which can be used to show that a ring is not CM, but does not show when it is CM, is to compute the Hilbert series of R.

Working from the definitions of the Hilbert series, and using that the Hilbert series of a tensor product of graded modules is the product of their respective Hilbert series, it is simple to show:

Proposition 2. *If a finitely generated, free* $\mathbb{C}[x_1, \ldots, x_n]$*-module* M *has generators* m_i*, where* $\deg m_i = d_i$*, and we assign* $\deg x_i = p_i$*, then the Hilbert series* $h_M(t)$ *of* M *is*

$$h_M(t) = \frac{t^{d_1} + \ldots + t^{d_n}}{(1 - t^{p_1}) \ldots (1 - t^{p_n})}.$$

Thus if we compute the Hilbert series of R, then if $(1 - t^{p_1}) \ldots (1 - t^{p_n}) h_A(t)$ has any negative coefficients, then A cannot be free over $\mathbb{C}[P_2, \ldots, P_r]$, which is equivalent to it not being CM by Corollary 1. Note that it is theoretically possible for this polynomial to have non-negative coefficient, yet A not be free.

This method was used to show that R is not CM for $\lambda = (4, 4, 2, 1)$, $(4, 4, 3, 1), (4, 4, 3, 2)$ and $(4, 1, 1, 1, 1)$ among others.

4.0.4. *Taking fibers.* The case $\lambda = (4, 3, 3, 1)$ was of interest, since computing its projective dimension was intractable, and the above method of computing its Hilbert series yielded $t^{13} + t^{12} + 2t^{11} + 2t^{10} + t^9 + t^8 + t^7 + t^6 + t^5 + 1$, suggesting that it may have been CM. Steven Sam suggested an additional method to resolve cases left indefinite by the previous method of looking for negative coefficients in the Hilbert series; we describe his method here. First, we recall another definition from commutative algebra:

Definition 10. For a ring R, an R-module M is called *flat* if, for all R-modules K, L and injective R-homomorphism $\varphi : L \to N$, then $\varphi \otimes \mathbb{1}_M : L \otimes_A M \to N \otimes_A L$ is injective.

Alternatively, since the tensor product is right-exact, for every short exact sequence $0 \to L' \to L \to L'' \to 0$ the sequence $0 \to L' \otimes M \to L \otimes M \to L'' \otimes M \to 0$ is exact.

In general, free modules are flat, and it is known that finitely generated flat modules over a local ring are free. So, if we show that R is flat over $\mathbb{C}[P_2, \ldots, P_r]$, we get that R is free, which we showed by Corollary 1 is equivalent to CM-ness.

Referring to section 13.7.4 in [Vak], given a prime ideal $\mathfrak{p} \subset A$ we can define the rank of R at p to be the dimension of the vector space $R_\mathfrak{p}/\mathfrak{m}R_\mathfrak{p} = R_\mathfrak{p} \otimes \kappa(\mathfrak{p})$, over the field $\kappa(\mathfrak{p}) = A_\mathfrak{p}/P_\mathfrak{p}$, and note that this is the rank of $R_\mathfrak{p}$ as an $A_\mathfrak{p}$-module.

As in exercise 13.7.J of the same reference, this defines an upper semi-continuous function on $\operatorname{Spec} A$, which by interpreting this condition in terms of the Zariski topology implies that if the rank at (0) equals the rank at $\mathfrak{m}$, then rank is constant. By the following, R is flat hence free:

Theorem 4.0.4 (13.7.K, [Vak])**.** *If M is a finitely generated module over the coordinate ring of a variety, then its rank is constant iff M is flat.*

So, we take $M = R$, and thus want the rank of $R \otimes_A \mathbb{C}(P_2, \ldots, P_r)$ and $R/\mathfrak{m}R$ to be the same. Alas, it is even quicker to compute the Hilbert series of $R/\mathfrak{m}R$, and we thus showed that R is CM for $\lambda = (4, 3, 3, 1)$.

4.1. **Flat families.** We now look at families of partitions, such as $\lambda = (a, \ldots, a, 1, \ldots, 1)$ where there are r copies of a and s of 1. Doing this both allows us to prove that for all but finitely many a, each partition in this family is CM, and gives us tools to compute the exceptional set for which λ is not CM, which we denote $B(r, s)$.

For a given $a \in \mathbb{C}$ we can consider $R_a = \langle P_i \rangle$; we also consider the ring which we denote $R[a] = \mathbb{C}[a, P_2, \ldots]$, where the a in the definition of the P_i is now a variable. We let X be the variety whose coordinate ring is $R[a]$. We have a map $\varphi : X \to \mathbb{A}^1$ by projecting onto a. Then R_a is the fiber of φ at $a \in \mathbb{A}^1$.

We want to find some finite $S \subset \mathbb{A}^1$ such that $\varphi^{-1}(\mathbb{A}^1 - S)$ is a flat family over $\mathbb{A}^1 - S$. For $S = \emptyset$ this means that $R[a]$ is flat over $\mathbb{C}[a]$, and in general removing the points $\{s_1, \ldots, s_n\} = S$ corresponds to taking the ring $R[a][\frac{1}{a-s_1}, \ldots, \frac{1}{a-s_n}]$ and asking whether it is flat over $\mathbb{C}[a]$. By the upper-semicontinuity of rank as discussed above, we can always remove a proper closed subset of $\mathbb{A}^1$, which is exactly just a finite set S, to attain a flat family.

Now it is known that the CM property is an open condition for flat families, i.e. if R_a is CM and $a \notin S$ then for some neighborhood U of a we have R_b is CM for $b \in U$; this means that if a single R_a is CM, then almost all are.

Well Theorem 2.0.2 tells us that R_a is CM for all integral $a > r$, hence for infinitely many a. Thus whatever finite set S we remove to get a flat family, infinitely many (hence at least one) of these R_a remain in the family. Thus we have proven:

Theorem 4.1.1. *Almost all such R_a are CM.*

4.1.1. *Detecting non-CM R_a.* Our computations, which used the projective dimension method described first, showed that $\lambda = (3, 1, 1, 1)$ and $(3, 3, 2, 2, 2)$ do not yield CM rings, so $3 \in B(3, 1), \frac{3}{2} \in B(2, 3)$. To calculate these $B(r, s)$ is to answer the question of whether R_a is CM or not completely, on the given families of λ, hence it is directly of interest to us.

To this end, we have an empirical method again for answering this question. By Theorem 9.9 in [Har], we know that the Hilbert polynomial of R_a is constant, for all a representing the flat family which we constructed. Therefore there is a generic Hilbert series, which is easy to determine after computing the Hilbert series for a few integral a. For example, for the family $(a, a, 1, 1)$, after multiplying through by $(1-t^2)(1-t^3)(1-t^4)$, we get $1+t^5+t^6+t^7+t^8+t^{10}$ for $a = 3, 4, \ldots, 50$. While we have not proven that this is the generic Hilbert series for this family, we presume that it is.

Now, this polynomial tells us which degrees to pick generators in, and once we have picked them we can write a general expression for how to generate higher degree P_i; for example, in this case we must have $P_{11} = (\alpha_1 P_2^4 P_3 + \alpha_2 P_2 P_3^2 + \alpha_3 P_4^2 P_3 + \alpha_4 P_2^2 P_3 P_4) + (\alpha_5 P_2^3 + \alpha_6 P_3^2 + \alpha_7 P_2 P_4) P_5 + \alpha_8 P_2 P_3 P_6 + (\alpha_9 P_2^2 + \alpha_{10} P_4) P_7 + \alpha_{11} P_3 P_8$. We can then solve for α_i as rational functions in a, and the values outside of S for which R_a is not CM will always appear as roots in the denominator of at least one α_i.

To state this formally:

Proposition 3. *If the ring R for a given λ has the generic Hilbert series $1 + a_1 t + \ldots + a_m t^m$, with nonnegative integer coefficients, and is generated by a finite subset S which consists of a_i elements of degree i, for $1 \le i \le m$, then R is CM.*

Proof. Let M be the free graded A-module, with generators in the same degrees as the chosen basis elements, and take the map $\varphi : M \to R_a$ which maps the free elements of M to the generators of R_a. Assuming that these elements generate R_a, φ must be a surjection. Then, since M and R_a have the same Hilbert series (the generic one), M_d and $(R_a)_d$ have the same dimension for all degrees d. This tells us that φ is an isomorphism and thus R_a is free over A, hence CM. □

If a particular value a_0 of a and a particular choice of basis which has a solution for the α_i (in the notation above) exist, such that a_0 is not a root of any of the denominators of α_i, we know that R_a must be CM, provided that the family is flat at a_0. We can simply plug $a = a_0$ into the α_i, which shows that the given basis generates R_a. This allows us to compute a finite set of a outside of which R_a must be CM.

5. Results

Let us collect the results of our various computations here:

5.1. **Projective dimension calculations.** Here are the results about the ring of invariants R for various λ, for which it was computationally viable to compute the projective dimension to verify CM-ness. We noted that every curve is CM, so all $|\lambda| = 2$ are CM cases. We also don't list λ of the form $(a, \ldots, a, 1, \ldots, 1)$ since Theorem 2.0.2 gives a complete answer for this type.

First we list the cases for $|\lambda| = 3$. Since all $|\lambda| = 2$ give CM rings, we never have occasion to use Proposition 1, so the only λ we have already accounted for are those of the form $(a, a, 1)$ and $(a, 1, 1)$.

λ	CM	λ	CM	λ	CM	λ	CM
321	false	322	true	332	true	421	false
431	true	432	false	433	true	443	true
521	false	522	true	531	false	532	true
533	true	541	true	542	false	543	false
544	true	552	true	553	true	554	true
621	false	631	false	632	false	641	false
643	false	651	true	652	false	653	false
654	false	655	true	665	true	721	false
722	true	731	false	732	false	733	true
741	false	742	false	743	true	744	true
751	false	752	true	753	false	754	false
755	true	761	true	762	false	763	false
764	false	765	false	766	true	772	true
773	true	774	true	775	true	776	true
821	false	831	false	832	false	833	true
841	false	843	false	851	false	852	true
853	true	854	false	855	true	861	false
863	false	865	false	871	true	872	false
873	false	874	false	875	false	876	false
877	true	883	true	885	true	887	true
921	false	922	true	932	false	941	false
942	false	943	false	944	true	951	false
952	false	953	false	954	true	955	true
961	false	962	false	964	false	965	false
971	false	972	true	973	false	974	false
975	false	976	false	977	true	981	true
982	false	983	false	984	false	985	false
986	false	987	false	988	true	992	true
994	true	995	true	997	true	998	true

Now let us list the values for which $|\lambda| = 4$ which we computed; we know that if the ring of invariants is not CM, then the whole ring R_λ is not CM, so we can use Proposition 1 and the previous table of $|\lambda| = 3$ data to eliminate some cases. We do not list these redundant cases (for example, the entire ring for $\lambda = 3211$ mustn't be CM, since it isn't for

$\lambda = 321$, since the ring of invariants isn't, as indicated by the above table).

λ	CM	λ	CM
3321	false	3222	true
3322	true	3332	true

For $\lambda_1 = 4$, let us give the time, in seconds, the computation ran for as well, to give some indication of how difficult these computations get. We also make note of cases which did not terminate.

λ	CM	Time (s)	λ	CM	Time (s)
4211	false	29.7	4311	false	56.2
4221	false	75.8	4321	–	∞
4331	–	∞	4431	–	∞
4322	false	62.2	4333	true	0.94
4433	true	3.8	4443	true	0.96

For $\lambda_1 = 4$, let us give the time, in seconds, the computation ran for as well, to give some indication of how difficult these computations get. We also make note of cases which did not terminate.

λ	CM	Time (s)	λ	CM	Time (s)
4211	false	29.7	4311	false	56.2
4221	false	75.8	4321	–	∞
4331	–	∞	4431	–	∞
4322	false	62.2	4333	true	0.94
4433	true	3.8	4443	true	0.96

Most cases after this with three or more unique parts do not terminate.

5.2. **Hilbert series calculations.** As we argued before, given our graded ring of invariants R if we consider $(1-t^2)\dots(1-t^r)h_R(t)$, since CM-ness of R is equivalent to freeness as a $\mathbb{C}[P_2,\dots,P_r]$ module, if R is CM then the given polynomial must have nonnegative coefficients. We list here some λ for which we proved that R is not CM by exhibiting a negative coefficient in this polynomial. We abbreviate some polynomials once we have exhibited a negative coefficient, so they take up a single line:

λ	$h_M(t) \cdot (1-t^2)(1-t^3)(1-t^4)$
4321	$t^{28}+t^{27}+t^{26}-4t^{24}-\ldots+t^9+t^8+t^7+t^6+t^5+1$
4421	$t^{22}-t^{19}-2t^{18}-\ldots+t^9+t^8+t^7+t^6+t^5+1$
4431	$t^{22}-t^{19}-t^{18}-\ldots+t^9+t^8+t^7+t^6+t^5+1$
4432	$t^{22}-t^{19}-2t^{18}-\ldots+t^9+t^8+t^7+t^6+t^5+1$
5322	$t^{22}-t^{19}-2t^{18}-\ldots+t^9+t^8+t^7+t^6+t^5+1$
5532	$t^{20}-t^{17}-2t^{16}+2t^{13}+2t^{12}+2t^{11}+2t^{10}+t^9+t^8+t^7+t^6+t^5+1$
5541	$t^{20}-t^{17}-2t^{16}+2t^{13}+2t^{12}+2t^{11}+2t^{10}+t^9+t^8+t^7+t^6+t^5+1$
5542	$t^{22}-t^{19}-2t^{18}-\ldots+t^9+t^8+t^7+t^6+t^5+1$
5543	$t^{22}-t^{19}-2t^{18}-\ldots+t^9+t^8+t^7+t^6+t^5+1$

5.3. **Generating higher degree P_i.** As in the previous section, we noted that if we took a set of ostensible generators of the degrees indicated by the Hilbert polynomial, and showed that they generated the family via rational functions of a, then for all values of a but the roots of the denominators, R_a is generated by their specialization (i.e. plugging a in). We collect here some empirical evidence that this holds, in the form of rational expressions for some of the lower degree polynomials in terms of lower generators.

Let us note, additionally, that for families such as $(a+1, a, 1, 1)$ and polynomials such as P_{14}, which we do not list here since these expressions are so large (see [Bro] for more data, specifically `polynomial_generator_expressions.txt`), these systems of coefficients are very overdetermined, i.e. on the order of 20 variables and over 200 equations. While empirical, the existence of any solutions at all over $\mathbb{C}(a)$ indicates to us that there is something more happening here.

To begin, the family $(a, 1, 1)$ has generic Hilbert polynomial $1+t^4+t^5$, hence we pick generators $1, P_4, P_5$ for R over $\mathbb{C}(a)[P_2, P_3]$. We then get the following expressions for P_6, P_7:

$$P_6 = \frac{3a^2-3a+2}{a(2+3a+a^2)}P_3^2 - \frac{5-3a+a^2}{2(2+3a+a^2)}P_2^3 + \left(\frac{1}{2}+\frac{5-3a+a^2}{2+3a+a^2}\right)P_2P_4$$

$$P_7 = -\frac{a^3-a^2+5a+2}{2a(2+3a+a^2)}P_2^2P_3 + \frac{a^2+5}{2+3a+a^2}P_2P_5 + \frac{a^3+5a^2-2a+4}{2a(2+3a+a^2)}P_3P_4$$

The choice of generators P_6, P_7 is the only nontrivial one we can make, since we are working over $\mathbb{C}[P_2, P_3]$, and hence the roots of the denominators above are roots of all denominator expressions, i.e. are all *bad values* of a (belonging to the set $B(1,2)$). Moreover, for P_{10}, we get

$$P_{10} = -\frac{5a^9+30a^8+30a^7-30a^6+305a^5-476a^4+384a^3-752a^2+32a-32}{4(a-2)a^2(2+3a+a^2)^3}P_2^2P_3^2$$
$$+\frac{3(a^6-a^5-12a^4+23a^3-69a^2+50a-100)}{8(2+3a+a^2)^3}P_2^5$$
$$+\frac{8-4a+10a^2-5a^3+10a^4+5a^5}{2a^2(2+3a+a^2)^2}P_3^2P_4$$
$$+\left(\frac{1}{8}-\frac{3(-100+50a-69a^2+23a^3-12a^4-a^5+a^6)}{4(2+3a+a^2)^3}\right)P_2^3P_4$$
$$+\frac{-64+32a-80a^2+35a^3-5a^4+5a^5+5a^6}{2(a-2)a(2+3a+a^2)^2}P_2P_3P_5$$

Of note, we see that the roots of these denominators are $a=-2,-1,0,2$, and we note that for $(2,1,1)$, R is not CM. We conjecture that these are all the bad values, i.e. $B(1,2)=\{-2,-1,0,2\}$, and simliarly for the families $(a,a,1)$, $(a,1,1,1)$, $(a,a,1,1)$, $(a,1,1,1,1)$, and $(a,a,1,1,1)$, the data of which is in the aforementioned resource.

5.4. **Further results.** More data, including the source codes for all the programs written here, is all available on Github at [Bro].

6. Future work and conjectures

Much of the work in this paper is empirical, and applies to the ring of invariants R of the larger ring R_λ, hence there is much to be done still. Let us describe some ideas for future research.

6.1. **Generic CM-ness of** $(a+1,a,1,1)$**.** Using Theorem 2.0.2, we were able to prove that families of the form $(a,\dots,a,1,\dots,1)$ are CM generically, using arguments from algebraic geometry about flat families. The next most simple type of family to consider, for which there is no corresponding theorem from Cherednik algebra theory to furnish infinitely many values of a for which the partition is CM (giving us at least one such value of a when we remove a finite set S to get our flat family), is $(a+1,a,1,\dots,1)$. Note that if we could prove this is CM for infinitely many a, the same argument tells us it is CM for almost all a.

Conjecture 1. *The family* $(a+1,a,1,\dots,1)$ *is generically CM.*

In the case of $(a+1,a,1)$, calculations previously done by Steven Sam suggest that R is generated for generic a by $P_2,\dots,P_7$. Further calculations show that for $4\le a\le 50$ the partition $(a+1,a,1,1)$ has the generic Hilbert series $t^{13}+t^{12}+2t^{11}+2t^{10}+t^9+t^8+t^7+t^6+t^5+1$, and calculations in the file `polynomial_generator_expressions.txt` in [Bro] show that $P_i, i \geq 14$ seem to be generated by a basis of elements of the appropriate degree; proving this would show that the family is generically CM, by previous arguments.

Another extrapolation of the data collected is the following conjecture, which may be helpful in proving the above one:

Conjecture 2. *The ring of invariants R is CM for $\lambda = (a+1, a, 1, \dots, 1)$, where there are p 1s, if and only if R is CM for $(a, a, 1, \dots, 1)$, where there are $p+1$ 1s.*

This is consistent with all the data we have collected so far, as well as for $\lambda = 54111$, 65111, 7433, and 5322.

6.2. **The sets $B(r, s)$.** We know that the sets $B(r, s)$ as defined above are finite, and that they consist of the values of a which are roots of a denominator in all choices of basis for the coefficients in $C(a)$ to generate higher degree P_i, using lower degree ones as indicated by the Hilbert series. We can show from Theorem 2.0.2 that:

Proposition 4. $B(r, s) \subset B(r+1, s) \cap B(r, s+1)$

Proof. It is immediate that $B(r, s) \subset B(r+1, s)$ from Theorem 2.0.2. Now, if a is not an integer then the same method as in Proposition 1 gives us that $a \in B(r+1, s)$; if a is an integer, then Theorem 2.0.2 tells us that this is equivalent to saying that $a \leq s$, which immediately tells us that $a \in B(r+1, s)$ as well. □

However, not much else is known about these $B(r, s)$. In particular, we would like to know the following:

Conjecture 3. *The sets $B(r, s)$ consist of rational values.*

The computations to do with expressions for higher degree polynomials in terms of generators give us evidence for this claim. Let us here list the roots of denominators in these expressions, for each family, to see that at least for low degree P_i all the coefficients in $\mathbb{C}(a)$ of the generators have only rational roots in their denominators:

Family	Roots
$(a, 1, 1)$	$-2, -1, 0, 2$
$(a, a, 1)$	$-1, -\frac{1}{2}, 0$
$(a+1, a, 1)$	$-2, -1, -\frac{1}{2}, \frac{1}{2}, 2$
$(a, a, 1, 1)$	$-2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 2$
$(a+1, a, 1, 1)$	$-3, -2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, 3$
$(a, 1, 1, 1, 1)$	$-4, -3, -2, -1, 0, 2, 3, 4$
$(a, a, 1, 1, 1)$	$-3, -2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2}, 2, 3$

6.3. **The isotypic component of the reflection representation.** We analyzed the ring of invariants R of the larger ring R_λ for most of this paper. Interpreting the ring R_λ as a certain representation of S_n, this corresponds to analyzing the isotypic component of the trivial representation of S_n. The next simplest case to analyze is accordingly the isotypic component of the reflection representation of S_n.

Let us describe this for $|\lambda| = 3$, the general case being entirely analogous. Recall that for $\lambda = (p, q, r)$ we set $a = \frac{p}{r}, b = \frac{q}{r}$ and that the ring of invariants can be described by $R = \langle P_i \rangle$ where $P_i = ax^i + by^i + (-1)^i(ax+by)^i$. Then we can describe the isotypic component of the reflection representation as the submodule $M \subset R \oplus R$ generated by $R_j = (x^j + (ax+by)^j)u + (y^j + (ax+by)^j)v$ with $j = 1, 2, \ldots$, where $u = (1, 0), v = (0, 1)$, and we define $\deg u = \deg v = 0$.

Given this expression for the module M, we wish to modify much of the code and methods described above to this case.

7. Acknowledgements

The math described in this paper was done as a part of the MIT SPUR program; as such, the author would like to thank Dr. Slavovitch, Prof. Etingof, and Prof. Jerison for running such an effective program. The author would especially like to thank David Corwin for his hours explaining the arguments we outlined in this paper, often several times. He would also like to thank Steven Sam, for taking the time out to explain much of his code which this work was based on, and for numerous suggestions of ways to compute various quantities used in calculating the results of this project. Finally, the author would especially like to thank his parents for supporting him to pursue SPUR, in spite of there being more lucrative and less fulfilling ways to spend his time.

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