An Inequality for the Antiferromagnetic Potts Model

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Abstract

In this paper, we study a graph-theoretic inequality on a general Potts model partition function $Z(G, \Omega_V)$, allowing weights on the vertices and colors. We first discuss a purely algebraic inequality, which we prove for integral and sufficiently large real exponents. From this algebraic inequality, we deduce several local graph-theoretic inequalities. For the case $\beta = 0$, which counts the number of weighted proper colorings, we use these local inequalities to prove, by induction, an upper bound on $Z(G, \Omega_V)$ in terms of bipartite graphs.

Summary

We explore a mathematical inequality arising from a physical model of interacting particles represented by a graph. It is still an open problem as to which graph maximizes the value of a certain quantity. This conjecture can be extended to a more general inequality, removing constraints on our objects of interest. In this paper, we study the general inequality under certain assumptions, namely that one parameter β is zero. We translate an algebraic result into inequalities on graphs, proving the general conjecture in these cases.

1 Introduction

The Potts model is a model in statistical mechanics describing interacting particles and their spins. Mathematically, we depict this as a graph on the particles, coloring the vertices to represent the spins. While this model has important physical consequences, it also raises interesting questions in extremal graph theory.

We explore a graph-theoretic inequality arising from the Potts model. Let $G = (V, E)$ be a graph. Fix $\beta \in [0, 1]$ and a positive integer q. We define the *Potts model partition function* to be

$$Z(G, \beta) = \sum_{\phi \in [q]^{|V|}} \beta^{m(\phi)},$$

where $m(\phi)$ denotes the number of monochromatic edges of the q-coloring ϕ. Galvin and Tetali [1] made the following conjecture:

Conjecture 1.1. *Over all d-regular graphs G, the quantity*

$$Z(G, \beta)^{1/|V(G)|}$$

is maximized when $G = K_{d,d}$.

Their conjecture can be extended, dropping the condition that G is regular.

Conjecture 1.2. *For any graph G with no isolated vertices, the inequality*

$$Z(G, \beta) \leq \prod_{uv \in E} Z\left(K_{d_u, d_v}, \beta\right)^{\frac{1}{d_u d_v}} \tag{1}$$

holds (here $d_v = \deg v$).

We consider a generalized version of Conjecture 1.2 where additional weights are permitted on the vertices and colors. Let $G = (V, E)$ be a graph with no isolated vertices. For each $v \in V$, consider a measure space $\Omega_v = ([q], \mu_v)$, where μ_v is a function from $[q]$ to $\mathbb{R}_{\geq 0}$. Also, let Ω_V to be the product measure space $\left([q]^{|V|}, \mu_V\right)$.

Definition 1.1. Define the function

$$Z(G, \Omega_V) = \int_{\Omega_V} \beta^{m(\phi)} \, \mathrm{d}\phi$$

where $m(\phi)$ is the number of monochromatic edges of a coloring $\phi \in [q]^{|V|}$.

Definition 1.2. Given two measure spaces A and B on $[q]$, define the expression

$$(A, B; s, t) = Z(K_{s,t}, A^s \times B^t).$$

In this paper, we explore the following conjecture, which generalizes Conjecture 1.2 to arbitrary measure spaces.

Conjecture 1.3. *Let G be a graph with no isolated vertices. Define a measure space Ω_v for each $v \in V$. Then*

$$Z(G, \Omega_V) \leq \prod_{uv \in E} (\Omega_u, \Omega_v; d_v, d_u)^{\frac{1}{d_u d_v}}.$$

We prove certain special cases of this conjecture assuming a purely algebraic inequality.

Conjecture 1.4. *Let q and c be positive integers, and $p_1, \ldots, p_q$ be nonnegative real numbers with sum 1. For any nonnegative real numbers $x_1, \ldots, x_q$ and real number $t \geq 1$, the inequality*

$$\sum_{1 \leq i_1, \ldots, i_c \leq q} \left(\sum_{\substack{1 \leq j \leq q \\ j \neq i_1, \ldots, i_c}} p_j \right)^t \prod_{k=1}^{c} x_{i_k} \geq \left(\sum_{i=1}^{q} \left(\sum_{\substack{1 \leq j \leq q \\ j \neq i}} p_j \right)^t x_i \right)^c \tag{2}$$

holds.

In Section 2, we prove Conjecture 1.4 when t is an integer or $t \geq q - 1$. This inequality is related to 1.3 by the following theorem.

Theorem 1.5. *If Conjecture 1.4 is true, then Conjecture 1.3 is true for $\beta = 0$.*

The classical inequality (1) has been proven for several special cases. Galvin and Tetali [1] proved inequality (1) for bipartite G, Davies et al. [2] proved inequality (1) for 3-regular graphs and Davies [3] extended the proof to 4-regular graphs using computer-assisted techniques, thus answering Conjecture 1.1 for the cases $d = 3, 4$. More recently, Sah et al. [4] proved inequality (1) for triangle-free G. We build off of their ideas, considering the quantity $Z(G, \Omega_V)$ instead of $Z(G, \beta)$.

The paper is structured as follows. In Section 2, we discuss proofs of Conjecture 1.4 under certain assumptions. In Section 3, we use these algebraic inequalities to prove local inequalities on graphs. These local inequalities are then used to prove Theorem 1.5 in Section 4.

2 Algebraic Inequalities

We offer a proof of Conjecture 1.4 for two special cases. First, we restrict t to be an integer.

Theorem 2.1. *Conjecture 1.4 holds when t is a positive integer.*

Proof. Suppose we expand

$$\left(\sum_{\substack{1\le j\le q\\ j\neq i_1,\dots,i_c}} p_j\right)^t$$

with the multinomial theorem. Consider the coefficient of the term $p_{\ell_1}^{e_1}\cdots p_{\ell_n}^{e_n}$ in the expansion of the left-hand side, where the exponents $e_1,\dots e_n$ are positive integers with sum t. This coefficient is

$$\binom{t}{e_1,\dots,e_n}\sum_{\substack{1\le i_1,\dots,i_c\le q\\ i_a\neq \ell_b}}\left(\prod_{k=1}^{c} x_{i_k}\right)=\binom{t}{e_1,\dots,e_n}\left(\sum_{\substack{1\le i\le q\\ i\neq \ell_1,\dots,\ell_n}} x_i\right)^c.$$

It follows that the left-hand side of (2) can be written as

$$\sum\binom{t}{e_1,\dots,e_n}\left(\sum_{\substack{1\le i\le q\\ i\neq \ell_1,\dots,\ell_n}} x_i\right)^c p_{\ell_1}^{e_1}\cdots p_{\ell_n}^{e_n}$$

where the sum is over all monomials $p_{\ell_1}^{e_1}\cdots p_{\ell_n}^{e_n}$ of degree t. We use a similar argument for the right-hand side of (2) and rewrite the desired inequality as

$$\sum\binom{t}{e_1,\dots,e_n}\left(\sum_{\substack{1\le i\le q\\ i\neq \ell_1,\dots,\ell_n}} x_i\right)^c p_{\ell_1}^{e_1}\cdots p_{\ell_n}^{e_n}\ge\left(\sum\binom{t}{e_1,\dots,e_n}\left(\sum_{\substack{1\le i\le q\\ i\neq \ell_1,\dots,\ell_n}} x_i\right) p_{\ell_1}^{e_1}\cdots p_{\ell_n}^{e_n}\right)^c.$$

This follows by weighted Jensen's inequality and the convexity of $x\mapsto x^c$, since the sum of the weights is

$$\sum \binom{t}{e_1, \ldots, e_n} p_{\ell_1}^{e_1} \cdots p_{\ell_n}^{e_n} = \left(\sum_{i=1}^{n} p_i\right)^t = 1.$$

□

Our strategy of expanding the sums raised to the power of t can be extended to sufficiently large real numbers t. We generalize the multinomial expansion with noninteger powers as follows. For each $n \in \mathbb{N}$, define the symmetric function $f_n : \mathbb{R}^n \to \mathbb{R}$ by

$$f_n(x_1, \ldots, x_n) = \sum_{S \subseteq [n]} \left(\sum_{i \in S} x_i\right)^t (-1)^{n-|S|},$$

where the empty sum is zero. It can be checked that

$$(x_1 + \cdots + x_n)^t = \sum_{\{s_1, \ldots, s_k\} \subseteq [n]} f_k(s_1, \ldots, s_k). \tag{3}$$

This is our analogue of the multinomial expansion.

Lemma 2.2. *If $t \geq n-1$, then $f_n(x_1, \ldots, x_n) \geq 0$ for any $x_1, \ldots, x_n \geq 0$.*

Proof. We may rewrite

$$f_n(x_1, \ldots, x_n) = \int_0^{x_1} \cdots \int_0^{x_n} t(t-1)\cdots(t-n+1)(y_1 + \cdots + y_n)^{t-n} \mathrm{d}y_1 \cdots \mathrm{d}y_n,$$

as the n^{th} derivative of x^t is $t(t-1)\cdots(t-n+1)x^{t-n}$. Since $t \geq n-1$, the integrand is always nonnegative, and the conclusion follows. □

Because the terms of our modified multinomial expansion are nonnegative, we may use the same strategy as the proof of Theorem 2.1.

Theorem 2.3. *Conjecture 1.4 holds for all $t \geq q-1$.*

Proof. Let us expand

$$\left(\sum_{\substack{1 \leq j \leq q \\ j \neq i_1, \ldots, i_c}} p_j\right)^t$$

using equation (3). After expanding the left-hand side of (2), the coefficient of $f_k(p_{\ell_1}, \ldots, p_{\ell_k})$

is

$$\sum_{\substack{1\le i_1,\ldots,i_c\le q\\ i_a\ne \ell_b}}\left(\prod_{k=1}^{c} x_{i_k}\right)=\left(\sum_{\substack{1\le i\le q\\ i\ne \ell_1,\ldots,\ell_n}} x_i\right)^c.$$

Performing a similar expansion on the right-hand side of (2), we are left to show

$$\sum_{\{\ell_1\ldots,\ell_n\}\subseteq[q]}\left(\sum_{\substack{1\le i\le q\\ i\ne \ell_1,\ldots,\ell_n}} x_i\right)^c f\left(p_{\ell_1},\ldots,p_{\ell_n}\right)\ge\left(\sum_{\{\ell_1\ldots,\ell_n\}\subseteq[q]}\left(\sum_{\substack{1\le i\le q\\ i\ne \ell_1,\ldots,\ell_n}} x_i\right) f\left(p_{\ell_1},\ldots,p_{\ell_n}\right)\right)^c.$$

This follows by weighted Jensen's inequality and the convexity of $x\mapsto x^c$: by Lemma 2.2, the weights are nonnegative, and their sum is

$$\sum_{\{\ell_1,\ldots,\ell_n\}\subseteq[q]} f_n(\ell_1,\ldots,\ell_n)=(x_1+\cdots+x_q)^t=1.$$

□

3 Local Inequalities

We discuss several local inequalities needed for the proof of Theorem 1.5. These lemmas are analagous to the results of [4, Section 4].

Definition 3.1. Let $A=([q],\mu)$ be a measure space and $\mathbf{x}\subseteq[q]$. Then we define

$$A\ominus\mathbf{x}=([q],\mu')$$

where

$$\mu'(x)=\begin{cases}\mu(x) & x\notin\mathbf{x}\\ \beta\mu(x) & x\in\mathbf{x}\end{cases}.$$

Lemma 3.1. *Let A,B be measure spaces on $[q]$. For nonnegative integers k and $r\le s\le t$,*

$$(A,B;k,s)\le(A,B;k,r)^{\frac{t-s}{t-r}}(A,B;k,t)^{\frac{s-r}{t-r}}.$$

Proof. In general, we have the equality

$$(A,B;k,i)=\int_{A^k}|B\ominus\mathbf{x}|^i\mathrm{d}\mathbf{x}$$

for any nonnegative integer i. Applying this for $i=r,s,t$, the result follows by Hölder's inequality. □

Lemma 3.2. *Let B, C be measure spaces on $[q]$ and b, c be positive integers. Fix $y \in [q]$. If $\beta = 0$, then*

$$|C \ominus y|^{b-1} (B \ominus y, C; c-1, b-1) \geq |C|^{b-1}(B \ominus y, C \ominus y; c-1, b-1).$$

Proof. Rewrite the inequality as

$$\frac{1}{|C|^{b-1}} \int_{C^{b-1}} |B \ominus y \ominus \mathbf{x}|^{c-1} \mathrm{d}\mathbf{x} \geq \frac{1}{|C \ominus y|^{b-1}} \int_{(C \ominus y)^{b-1}} |B \ominus y \ominus \mathbf{x}|^{c-1} \mathrm{d}\mathbf{x}.$$

Treating $\mathbf{x} \in C^{b-1}$ as a random variable, the inequality can be expressed as

$$\mathbb{E}\left[|B \ominus y \ominus \mathbf{x}|^{c-1}\right] \geq \mathbb{E}\left[|B \ominus y \ominus \mathbf{x}|^{c-1} \middle| y \notin \mathbf{x}\right].$$

With this interpretation, the inequality is clearly true. $\square$

Conjecture 3.3. *Let A, B, C be measure spaces on $[q]$ and $a \geq b, c$ be positive integers. Then*

$$\int_{A^c} \prod_{i=1}^{c} (B \ominus x_i, C \ominus x_i; c-1, b-1)^{\frac{a}{b+c-2}} \mathrm{d}\boldsymbol{x} \leq \int_{A^c} |C \ominus \boldsymbol{x}|^{\frac{a(b-1)}{b+c-2}} \prod_{i=1}^{c} (B \ominus x_i, C; c, b-1)^{\frac{a(c-1)}{c(b+c-2)}} \mathrm{d}\mathbf{x}. \tag{4}$$

Lemma 3.4. *Conjecture 1.4 implies Conjecture 3.3 for $\beta = 0$.*

Proof. If $c = 1$ then equality always holds, so suppose that $c \geq 2$. After bounding the right-hand side below with Lemma 3.1, we are left to prove that

$$\begin{aligned} &\int_{A^c} |C|^{\frac{a(b-1)}{b+c-2}} \prod_{i=1}^{c} (B \ominus x_i, C \ominus x_i; c-1, b-1)^{\frac{a}{b+c-2}} \mathrm{d}\mathbf{x} \\ &\leq \int_{A^c} |C \ominus \mathbf{x}|^{\frac{a(b-1)}{b+c-2}} \prod_{i=1}^{c} (B \ominus x_i, C; c-1, b-1)^{\frac{a}{b+c-2}} \mathrm{d}\mathbf{x}. \end{aligned}$$

We apply Lemma 3.2 to the left-hand side, reducing the inequality to

$$\begin{aligned} &\int_{A^c} |C|^{\frac{a(b-1)}{b+c-2}} \prod_{i=1}^{c} \frac{|C \ominus x_i|^{\frac{a(b-1)}{b+c-2}}}{|C|^{\frac{a(b-1)}{b+c-2}}} (B \ominus x_i, C; c-1, b-1)^{\frac{a}{b+c-2}} \mathrm{d}\mathbf{x} \\ &\leq \int_{A^c} |C \ominus \mathbf{x}|^{\frac{a(b-1)}{b+c-2}} \prod_{i=1}^{c} (B \ominus x_i, C; c-1, b-1)^{\frac{a}{b+c-2}} \mathrm{d}\mathbf{x}. \end{aligned}$$

Suppose that $A = (a_1, \ldots, a_q)$ and $C = (c_1, \ldots, c_q)$ are the associated measures. We set $t = \frac{a(b-1)}{b+c-2}$ and

$$p_i = \frac{c_i}{|C|}, \ y_i = a_i (B \ominus x_i, C; c-1, b-1)^{\frac{a}{b+c-2}}$$

for each $1 \le i \le q$. Then it is enough to prove that

$$\left(\sum_{i=1}^{q} (1-p_i)^t \, y_i\right)^c \le \sum_{1\le i_1,\ldots,i_c\le q} \left(\sum_{\substack{1\le j\le q \\ j\ne i_1,\ldots,i_c}} p_j\right)^t \prod_{k=1}^{c} y_{i_k}.$$

This follows from Conjecture 1.4. □

Lemma 3.5. *Let A, B, C be measure spaces on $[q]$ and $a \ge b, c$ be positive integers. If inequality (4) holds, then*

$$\int_A (B\ominus x, C\ominus x; c-1, b-1)^{\frac{a}{b+c-2}} \mathrm{d}x \le (A, B; b, a)^{\frac{c-1}{b(b+c-2)}} (A, C; c, a)^{\frac{b-1}{c(b+c-2)}} (B, C; c, b)^{\frac{a(b-1)(c-1)}{bc(b+c-2)}}.$$

Proof. When $b = 1$ we have an equality, so assume that $b \ge 2$. We apply [4, Lemma 3.3], which implies that

$$\int_A (B\ominus x, C; c, b-1)^{\frac{a}{c}} \,\mathrm{d}x \le (A, B; b, a)^{\frac{1}{b}} (B, C; c, b)^{\frac{a(b-1)}{bc}}.$$

We are left to show

$$\int_A (B\ominus x, C\ominus x; c-1, b-1)^{\frac{a}{b+c-2}} \,\mathrm{d}x \le (A, C; c, a)^{\frac{b-1}{c(b+c-2)}} \left(\int_A (B\ominus x, C; c, b-1)^{\frac{a}{c}} \,\mathrm{d}x\right)^{\frac{c-1}{b+c-2}}.$$

After raising both sides to the power of c, we may rewrite the inequality as

$$\int_{A^c} \prod_{i=1}^{c} (B\ominus x_i, C\ominus x_i; c-1, b-1)^{\frac{a}{b+c-2}} \,\mathrm{d}\mathbf{x}$$
$$\le \left(\int_{A^c} |C\ominus \mathbf{x}|^a \,\mathrm{d}\mathbf{x}\right)^{\frac{b-1}{b+c-2}} \left(\int_{A^c} \prod_{i=1}^{c} (B\ominus x_i, C; c, b-1)^{\frac{a}{c}} \,\mathrm{d}\mathbf{x}\right)^{\frac{c-1}{b+c-2}}$$

where the x_i are the entries of the c-vector $\mathbf{x} \in A^c$. Applying Hölder's inequality to the right-hand side, it suffices to show

$$\int_{A^c} \prod_{i=1}^{c} (B\ominus x_i, C\ominus x_i; c-1, b-1)^{\frac{a}{b+c-2}} \,\mathrm{d}\mathbf{x}$$
$$\le \int_{A^c} |C\ominus \mathbf{x}|^{\frac{a(b-1)}{b+c-2}} \prod_{i=1}^{c} (B\ominus x_i, C; c, b-1)^{\frac{a(c-1)}{c(b+c-2)}} \mathrm{d}\mathbf{x}.$$

This follows from Lemma 3.4. □

4 Main Induction

In this section, we prove Theorem 1.5 by induction on $|V|$ in the same manner as the proof of [4, Theorem 4.1]. The base case $|V| = 0$ is trivial, so we only consider the inductive step.

Let Δ be the largest degree of a vertex in V, and fix a vertex w of degree Δ. Let V_k be the set of all vertices at a distance of exactly k from w, and E_{ij} be the set of edges with one endpoint in V_i and the other endpoint in V_j. In particular, $uv \in E_{ij}$ means that $u \in V_i$ and $v \in V_j$. We use $E_{\geq 2}$ to denote the set

$$\bigcup_{\min(i,j)\geq 2} E_{ij}.$$

Finally, let $I_1 \subseteq V_1$ be the set of neighbors of w with degree 1.

For each edge $uv \in E$, define a function $f_{uv} : \Omega_u \times \Omega_v \to \{0, 1\}$ by

$$f_{uv}(x_u, x_v) = \begin{cases} \beta & x_u = x_v \\ 1 & \text{otherwise} \end{cases}.$$

Also, for each $x_w \in [q]$ and $v \in V \setminus w$, define a new measure space $\Omega_v^{x_w} = ([q], \mu_v^{x_w})$ as

$$\mu_v^{x_w}(x) = \begin{cases} \beta\mu_v(x) & v \in V_1 \text{ and } x = x_w \\ \mu_v(x) & \text{otherwise} \end{cases}. \tag{5}$$

Let $G' = (V', E')$ be the subgraph of G induced by the vertex set $V \setminus w \cup I_1$. Then

$$\begin{aligned}
Z(G, \Omega_V) &= \int_{\Omega_V} \beta^{m(\phi)} \mathrm{d}\phi \\
&= \int_{\Omega_w} \int_{\Omega_{V\setminus w}^{x_w}} \beta^{m(\phi')} \mathrm{d}\phi' \mathrm{d}x_w \\
&= \int_{\Omega_w} \left(\int_{\Omega_{V'}^{x_w}} \beta^{m(\phi')} \mathrm{d}\phi' \right) \prod_{v \in I_1} \left(\int_{\Omega_v} f_{wv}(x_w, x_v)\ \mathrm{d}x_v \right) \mathrm{d}x_w \\
&\leq \int_{\Omega_w} \prod_{u'v' \in E'} (\Omega_{u'}^{x_w}, \Omega_{v'}^{x_w}; d_{v'}, d_{u'})^{1/(d_{u'}d_{v'})} \prod_{v \in I_1} \left(\int_{\Omega_v} f_{wv}(x_w, x_v)\ \mathrm{d}x_v \right) \mathrm{d}x_w,
\end{aligned}$$

where the final inequality follows from the inductive hypothesis on G'. It then suffices to show that

$$\int_{\Omega_w} \prod_{u'v' \in E'} (\Omega_{u'}^{x_w}, \Omega_{v'}^{x_w}; d_{v'}, d_{u'})^{1/(d_{u'}d_{v'})} \prod_{v \in I_1} \left(\int_{\Omega_v} f_{wv}(x_w, x_v) \mathrm{d}x_v \right) \mathrm{d}x_w \leq \prod_{uv \in E} (\Omega_u, \Omega_v; d_v, d_u)^{1/(d_u d_v)}.$$

Upon dividing by

$$\prod_{uv \in E_{\geq 2}} (\Omega_u, \Omega_v; d_v, d_u)^{1/(d_u d_v)},$$

which appears as a factor on both sides, it is enough to prove

$$\int_{\Omega_w} \prod_{uv\in E_{12}} (\Omega_u^{x_w}, \Omega_v^{x_w}; d_v, d_u-1)^{1/((d_u-1)d_v)} \prod_{uv\in E_{11}} (\Omega_u^{x_w}, \Omega_v^{x_w}; d_v-1, d_u-1)^{1/((d_u-1)(d_v-1))} \prod_{v\in I_1} |\Omega_v^{x_w}|\, dx_w$$

$$\leq \prod_{uv\in E_{01}\cup E_{11}\cup E_{12}} (\Omega_u, \Omega_v; d_v, d_u)^{(1/d_u d_v)}.$$

We distribute the factors from E_{01} on the right-hand side to the factors from E_{11} and E_{12}, obtaining

$$\int_{\Omega_w} \prod_{uv\in E_{12}} (\Omega_u^{x_w}, \Omega_v^{x_w}; d_v, d_u-1)^{\frac{1}{(d_u-1)d_v}} \prod_{uv\in E_{11}} (\Omega_u^{x_w}, \Omega_v^{x_w}; d_v-1, d_u-1)^{\frac{1}{(d_u-1)(d_v-1)}} \prod_{v\in I_1} |\Omega_v^{x_w}|\, \mathrm{d}x_w$$

$$\leq \prod_{v\in I_1} (\Omega_w, \Omega_v, 1, \Delta)^{\frac{1}{\Delta}} \prod_{uv\in E_{11}} (\Omega_w, \Omega_u, d_u, \Delta)^{\frac{1}{d_u(d_u-1)\Delta}} (\Omega_w, \Omega_v, d_v, \Delta)^{\frac{1}{d_v(d_v-1)\Delta}} (\Omega_u, \Omega_v, d_v, d_u)^{\frac{1}{d_u d_v}}$$

$$\cdot \prod_{uv\in E_{12}} (\Omega_w, \Omega_u, u, \Delta)^{1/(d_u(d_u-1)\Delta)} (\Omega_u, \Omega_v; d_v, d_u)^{1/(d_u d_v)}.$$

We now use Hölder's inequality with the weights

$$\sum_{uv\in E_{12}} \frac{1}{(d_u-1)\Delta} + \sum_{uv\in E_{11}} \left(\frac{1}{(d_u-1)\Delta} + \frac{1}{(d_v-1)\Delta}\right) + \sum_{v\in V_1} \frac{1}{\Delta} = 1$$

to bound the left-hand side above by

$$\prod_{v\in I_1} \left(\int_{\Omega_w} |\Omega_v^{x_w}|^{\Delta}\, \mathrm{d}x_w\right)^{\frac{1}{\Delta}} \prod_{uv\in E_{11}} \left(\int_{\Omega_w} (\Omega_u^{x_w}, \Omega_v^{x_w}, d_v-1, d_u-1)^{\frac{\Delta}{d_u+d_v-2}} \mathrm{d}x_w\right)^{\frac{d_u+d_v-2}{(d_u-1)(d_v-1)\Delta}}$$

$$\cdot \prod_{uv\in E_{12}} \left(\int_{\Omega_w} (\Omega_u^{x_w}, \Omega_v, d_v, d_u-1)^{\Delta/d_v} \mathrm{d}x_w\right)^{\frac{1}{(d_u-1)\Delta}}.$$

The first of these factors satisfies

$$\int_{\Omega_w} |\Omega_v^{x_w}|^{\Delta}\, \mathrm{d}x_w = (\Omega_w, \Omega_v; 1, \Delta).$$

Applying Lemma 3.5 with $a = \Delta$, $b = d_u$, and $c = d_v$, we obtain the following bound on the second factor.

$$\prod_{uv\in E_{11}} \left(\int_{\Omega_w} (\Omega_u^{x_w}, \Omega_v^{x_w}, d_v-1, d_u-1)^{\frac{\Delta}{d_u+d_v-2}} \mathrm{d}x_w\right)^{\frac{d_u+d_v-2}{(d_u-1)(d_v-1)\Delta}}$$

$$\leq \prod_{uv\in E_{11}} (\Omega_w, \Omega_u, d_u, \Delta)^{\frac{1}{d_u(d_u-1)\Delta}} (\Omega_w, \Omega_v, d_v, \Delta)^{\frac{1}{d_v(d_v-1)\Delta}} (\Omega_u, \Omega_v, d_v, d_u)^{\frac{1}{d_u d_v}}.$$

Finally, [4, Lemma 3.3] provides a bound on the third factor.

$$\prod_{uv\in E_{12}} \left(\int_{\Omega_w} (\Omega_u^{x_w}, \Omega_v, d_v, d_u - 1)^{\Delta/d_v} \mathrm{d}x_w \right)^{\frac{1}{(d_u-1)\Delta}}$$

$$\leq \prod_{uv\in E_{12}} (\Omega_w, \Omega_u; d_u, \Delta)^{\frac{1}{d_u(d_u-1)\Delta}} (\Omega_u, \Omega_v; d_v, d_u)^{\frac{1}{d_u d_v}}.$$

Multiplying these three final inequalities, we obtain the desired result. □

5 Conclusion

In this paper, we explore a generalized inequality for the Potts model partition function. Conditioned on the purely algebraic Conjecture 1.4, we develop local inequalities that hold for any graph. We use these local inequalities to prove Theorem 2.1 by induction, providing bounds on the general Potts model partition function for $\beta = 0$ as well as the classical partition function for small positive β.

While we have proved Theorem 2.1 and Theorem 2.3, it remains to prove Conjecture 1.4 for real (or alternatively rational) $t \in [1, q-1]$, which is necessary for the main induction. It would be interesting to prove Conjecture 1.4 for specific values of $t \geq 1$, which may perhaps allow the induction to work on graphs with small maximum degree. Additionally, we conjecture that Lemma 3.4 holds for arbitrary $\beta \in [0, 1]$, which would extend Theorem 1.5 to β in this interval. Our proof of Lemma 3.4 does not generalize; it fails for β close to 1. Another approach to this local inequality is needed to extend our results to more values of β.

However, our proofs of the local inequalities may be extended for small $\beta > 0$. To do this, it is necessary to analyze the equality cases of these inequalities for $\beta = 0$ and show that they can be extended to small $\beta > 0$, perhaps by a power series expansion around $\beta = 0$. Such a result could be used to prove Conjecture 1.2 for small β.

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References

[1] D. Galvin and P. Tetali. On weighted graph homomorphisms. In *Graphs, morphisms and statistical physics*, volume 63 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 97–104. Amer. Math. Soc., Providence, RI, 2004.

[2] E. Davies, M. Jenssen, W. Perkins, and B. Roberts. Extremes of the internal energy of the Potts model on cubic graphs. *Random Structures Algorithms*, 53(1):59–75, 2018.

[3] E. Davies. Counting proper colourings in 4-regular graphs via the Potts model. *Electron. J. Combin.*, 25(4):Paper No. 4.7, 17, 2018.

[4] A. Sah, M. Sawhney, D. Stoner, and Y. Zhao. A reverse Sidorenko inequality. *Invent. Math.*, 221(2):665–711, 2020.