# On the Sizes of Furstenberg Sets in Finite Fields

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## Abstract

A $(k,m)$-*Furstenberg set* in $\mathbb{F}_q^n$ is a set of points such that every $k$-dimensional subspace of $\mathbb{F}_q^n$ has a translation containing at least $m$ points of the set. We explore the question of estimating the size of the smallest $(k,m)$-Furstenberg set in $\mathbb{F}_q^n$, denoted by $K(q,n,k,m)$. We provide several general constructions for small Furstenberg sets which yield upper bounds on $K(q,n,k,m)$. In particular, we show that there is a universal constant $C$ such that for large $m$, $K(q,n,1,m) \leq Cq^{\frac{n-1}{2}} m^{\frac{n+1}{2}}$, which is not far from the known lower bounds. We show another upper bound that $K(p^l,n,1,p^{l-1}) \leq Cm^2$, constituting an optimal upper bound up to constants. We also generalize existing lower bounds when $k = \sqrt{q}$, constituting an improvement of the easy lower bound $Cm^2$. Finally, we suggest other methods to potentially obtain new lower bounds.



## Summary

A *Furstenberg set* is a set of points which contains a large number of points on a line in each direction. In this paper, we explore the question of how small a Furstenberg set can be. We provide constructions for small Furstenberg sets, yielding upper bounds for the size of the smallest Furstenberg set. We then generalize existing lower bounds in certain cases. Finally, we suggest other methods to potentially obtain new lower bounds.

# 1 Introduction

In 1917, Japanese mathematician Soichi Kakeya [1] first introduced the concept of a Kakeya set in $\mathbb{R}^n$ — a set of points containing a line segment of length 1 in every direction. The study of these sets led to the famous Kakeya conjecture, which states that every Kakeya set in $\mathbb{R}^n$ has Hausdorff dimension $n$.

The Kakeya conjecture is not only an intriguing and difficult question in its own right but also has many important connections to areas such as harmonic analysis. In fact, the restriction and local smoothing problems in the latter area are highly related to and would both imply the Kakeya conjecture [2].

In 1999, Tom Wolff [2] proposed an analogue of Kakeya sets over finite fields. Informally, a Kakeya set over a finite field is a set containing a line in each direction. Because we are working over finite fields, the number of directions and the size of any Kakeya set are finite.

Wolff also posed the finite field analogue of the Kakeya conjecture in $\mathbb{R}^n$, which asks for the smallest possible size of a Kakeya set in $\mathbb{F}_q^n$. This was completely proven by Zeev Dvir [3] via the *polynomial method*.

**Theorem 1.1** (Dvir [3])**.** *Let $K$ be a Kakeya set over $\mathbb{F}_q^n$. Then $|K| \geq C_n q^n$, where $C_n$ is a constant depending only on $n$.*

Because the entire set $\mathbb{F}_q^n$, which has size $q^n$, is a Kakeya set, Theorem 1.1 gives the minimal value of $|K|$ up to a constant. Dvir, Dhar, and Lund [4] have since shown that the minimal value of $|K|$ is in the interval $[\frac{q^n}{2^n}, \frac{q^n}{2^{n-1}}]$, which is accurate up to a factor of 2.

In this paper, we investigate a generalization of Kakeya sets in finite fields known as *Furstenberg sets*. In Section 2, we give the definition of Furstenberg sets in $\mathbb{F}_q^n$ and introduce our main question of estimating the minimal size of Furstenberg sets. Section 3 reviews the existing bounds on the smallest size of a Furstenberg set in a finite field. In Section 4, we provide several constructions giving new upper bounds on Furstenberg set sizes, and in Section 5 we generalize existing lower bounds. These constitute our main results. In Section 6 we show relations between Furstenberg bounds which could yield lower and upper bounds. Finally, in Section 7 we offer another potential approach towards progress in the Furstenberg problem.

## 2 Furstenberg Sets

A natural generalization of Kakeya sets arises from considering when the set almost contains a line in each direction, but misses some points. These are known as *Furstenberg sets*.

**Definition 2.1.** A $(1, m)$-*Furstenberg set* in $\mathbb{F}_q^n$ is a set $F$ such that for any line $l \subset \mathbb{F}_q^n$, there exists some $w \in \mathbb{F}_q^n$ such that $|(w + l) \cap F| \geq m$.

Because a line contains at most $q$ points, in Definition 2.1, we must have $m \leq q$. When $m = q$, $F$ is a Kakeya set.

The parameter 1 in Definition 2.1 is because a line is a 1-dimensional subspace of $\mathbb{F}_q^n$. By replacing lines with higher dimensional subspaces we obtain a generalization of $(1, m)$-Furstenberg sets.

**Definition 2.2.** A $(k, m)$-*Furstenberg set* in $\mathbb{F}_q^n$ is a set $F$ such that for any $k$-dimensional subspace $S \subset \mathbb{F}_q^n$, there exists some $w \in \mathbb{F}_q^n$ such that $|(w + S) \cap F| \geq m$.

We refer to shifts $w + S$ of $k$-dimensional subspaces used in Definition 2.2 as *$k$-flats.*

For Furstenberg sets we ask the analogous question to the Kakeya problem: how small can a $(k, m)$-Furstenberg set in $\mathbb{F}_q^n$ be? We give a notation for the answer to this question.

**Definition 2.3.** Denote by $K(q, n, k, m)$ the smallest possible size of a $(k, m)$-Furstenberg set in $\mathbb{F}_q^n$.

Our question is therefore to determine the value of $K(q, n, k, m)$.

## 3 Existing Bounds on the Size of Furstenberg Sets

There has been progress on the question of evaluating $K(q, n, k, m)$ in the form of asymptotic upper and lower bounds. The following theorem is due to Ellenberg and Erman [5].

**Theorem 3.1** (Ellenberg and Erman [5]). *Let $k$,$q$, and $m$ be positive integers with $q$ a prime power and $m \leq q^k$. Then there is a constant $C_{k,n}$ depending only on $k$ and $n$ such that*

$$K(q, n, k, m) \geq C_{k,n} m^{\frac{n}{k}}.$$

For $m = q$ and $k = 1$, this is equivalent to Theorem 1.1, up to a constant. In addition, note that a $(k, m)$-Furstenberg set is automatically $(l, m)$-Furstenberg for $l > k$, hence we expect weaker bounds for larger $k$. Theorem 3.1 is therefore a natural generalization of Theorem 1.1.

When $q$ is prime and $m = \sqrt{q}$, Ruixiang Zhang [6] has improved Theorem 3.1 by a small amount.

**Theorem 3.2** (Zhang [6]). *Let $q$ be prime. Then there is a constant $C_n$ depending only on $n$ such that*

$$K(q, n, 1, \sqrt{q}) \geq C_n q^{\frac{n}{2} + \Omega\left(\frac{1}{n^2}\right)}.$$

Here, for $X, Y > 0$, where $X$ and $Y$ are quantities that depend on $n$, $X = \Omega(Y)$ means that there exists some positive constant $C$ for which $X \geq CY$ for all $n$.

Interestingly, when $q$ is not prime, Theorem 3.2 is false. In fact, Wolff [2] showed that the Ellenberg-Erman bound is optimal up to a constant for $q = p^2$.

**Theorem 3.3** (Wolff [2]). *Let $q = p^2$ be the square of a prime. Then $K(q, 2, 1, p) \leq Cp^2$, where $C$ is an absolute constant.*

However, Wolff's upper bound does not apply to prime fields or all values of $m$. One upper bound for general $m$ was given by Zhang [6].

**Theorem 3.4** (Zhang [6]). *Let $T > 0$ be a real number. For sufficiently large primes $q$ depending on $T$ and any integer $0 < m < \frac{q}{T}$,*

$$K(q, n, k, Tm) \leq Cq^{\frac{n-1}{2}} m^{\frac{n+1}{2}},$$

*where $C$ depends only on $K$.*

# 4 Constructions for small Furstenberg sets

In this section, we provide upper bounds for $K(q, n, 1, m)$ by constructing small Furstenberg sets. Our construction method is through taking the union of several subsets of $\mathbb{F}_q^n$ with one fixed coordinate, similar to Wolff's approach in [2].

One might expect Theorem 3.4 to generalize easily for all finite fields and yield the bound

$$K(q, n, k, m) \leq Cq^{\frac{1}{2}} m^{\frac{3}{2}}, \tag{1}$$

but it turns out that non-prime fields result in complications which hinder a direct attempt to generalize (1). Our first theorem proves (1) for all prime power fields $\mathbb{F}_q$ when $n = 2$ and $m$ is large.

**Theorem 4.1.** *Let $q$ be a prime power and $q = p^l$, where $p$ is prime. Let $m$ be an integer satisfying $\frac{q}{p} \leq m \leq q$. Then $K(q, 2, 1, m) \leq Cq^{\frac{1}{2}} m^{\frac{3}{2}} + m$, where $C$ is a universal constant.*

*Proof.* Let $M$ be the smallest integer not less than $\frac{mp}{q}$. Consider $\mathbb{F}_q = \mathbb{F}_p(w)$ as a simple field extension of $\mathbb{F}_p$. By properties of field extensions, every element of $\mathbb{F}_q$ can be written as $\sum_{i=0}^{l-1} a_i w^i$ for $a_i \in \mathbb{F}_p$.

Set $a > 0$ to be the integer closest to $\sqrt{\frac{p}{M}}$ and $\overline{a} \in \mathbb{Z} \cap [\frac{p}{a}, \frac{p}{a} + 1]$. Set $S = \{0, 1, \ldots a - 1\}$ and $T = \{0, a, 2a, \ldots, \overline{a}a\}$. Then let $T'$ be the set of elements that can be written as $\sum_{i=0}^{l-1} b_i w^i$, where $b_0 \in T$ and $b_i \in \mathbb{F}_p$ for $i \geq 1$.

For every $t \in \mathbb{F}_q$ let

$$E_t = \{(t, j + (j+k)t) : j \in S, k \in T'\}.$$

Note $j + (j+k)t = t[k + \frac{t+1}{t}j]$. Let $\mathcal{T}$ be the set of $t$ satisfying $\frac{t+1}{t} = \sum_{i=0}^{l-1} a_i w^i$ for $a_0 \in \{0, a, 2a, \ldots, Ma\}$ and $a_i \in \mathbb{F}_p$ for $i \geq 1$. Then $|\mathcal{T}| = (M+1)p^{l-1} \geq m$, so we may let $\mathcal{T}'$ be any subset of $\mathcal{T}$ with $m$ elements. Also let $V$ denote a set of any $m$ points of the form $(t, z)$ for fixed $t$. The key claim is that $F = V \cup \bigcup_{t \in \mathcal{T}'} E_t$ is a $(1, m)$-Furstenberg set.

Indeed, it suffices to show that for any $v \in \mathbb{F}_q$, there are some $a, b$ such that the line $\mathcal{L} : y = a + vx$. Observe that we may set $v = s + t, s \in S, t \in T'$ and $a = s$. Then for each $j \in \mathcal{T}'$, the point $(i, s + (s+t)i) \in E_j$ hence there are at least $|\mathcal{T}'| \geq m$ points in $F \cap \mathcal{L}$.

Finally we show $|F| \leq q^{\frac{1}{2}} m^{\frac{3}{2}} + m$. Note that any element of $E_j$ can be written as a sum

$$\sum_{i=0}^{l-1} c_i w^i \tag{2}$$

where by the definition of $E_j$ and $\mathcal{T}$, $c_0$ is in the set $\{0, a, 2a, \ldots, (Ma + \overline{a})a\}$ and $c_i \in \mathbb{F}_p$ for $i \geq 1$. Thus $|E_j| \leq [(Ma + \overline{a} + 1)]p^{l-1} \leq C\sqrt{pM}p^{l-1}$ where $C$ is approximately equal to 1.

Therefore by Equation (2) and the definitions of $|F|$ and $M$ we conclude that

$$|F| \leq m + \sum_{i \in \mathcal{T}} |E_i| \leq Cp^{l-1}\sqrt{pM}m + m = Cq^{\frac{1}{2}} m^{\frac{3}{2}} + m \tag{3}$$

where $C$ is approximately 1. □

**Remark 4.1.** The upper bound given by Equation 3 is off by a factor of at most $\sqrt{p}$ from an easy lower bound [6] of $\max(m^2, q^{\frac{1}{2}} m)$. Therefore when $l \gg 1$ we obtain a fairly close to optimal construction for most $m$.

Using a similar construction technique, we obtain the following bound in the cases not covered by Theorem 4.1. When $m$ and $q$ satisfy certain conditions, this bound is actually stronger than the one given by inequality (1).

**Theorem 4.2.** *Let $p$ be a prime, $q = p^l$ and $t \in \mathbb{Z}_{\geq 0}$. If $\frac{q}{p^{2t+3}} \leq m \leq \frac{q}{p^{2t+1}}$, then $K(q, 2, 1, m) \leq p^{l-t-1}m + m$.*

*Proof.* Consider the field extension $\mathbb{F}_q = \mathbb{F}_p(w)$. Every element of $\mathbb{F}_q$ can be written as $\sum_{i=0}^{l-1} a_i w^i$ for $a_i \in \mathbb{F}_p$.

Set $S$ as the set of elements of $\mathbb{F}_q$ which can be written as $\sum_{i=0}^{t} a_i w^i$ for $a_i \in \mathbb{F}_p$ and $T$ as the set of elements of $\mathbb{F}_q$ which can be written as $\sum_{i=t+1}^{l-1} b_i w^i$ for $b_i \in \mathbb{F}_p$.

For every $t \in \mathbb{F}_q$ let

$$E_t = \{(t, j + (j + k)t) : j \in S, k \in T\}.$$

Note $j + (j + k)t = t[k + \frac{t+1}{t} j]$. Consider the set $\mathcal{T}$ of $t$ satisfying $\frac{t+1}{t} = \sum_{i=t+1}^{l-t-1} a_i w^i$ for some $a_i \in \mathbb{F}_p$. Then $|\mathcal{T}'| = p^{l-2t-1} \geq m$. Therefore we may let $\mathcal{T}'$ be a subset of $\mathcal{T}$ with at most $m$ elements, and let $V$ denote a set of any $m$ points of the form $(t, z)$ for fixed $t$. The key claim is that $F = V \cup \bigcup_{t \in \mathcal{T}'} E_t$ is a $(1, m)$-Furstenberg set.

Indeed, for any slope $v \in \mathbb{F}_q$, set $s \in S, t \in T, s + t = v$. Let $\mathcal{L} : y = s + (s + t)x$. Then for every $j \in \mathcal{T}'$, the point $(j, s + (s + t)j) \in E_j$. Therefore $\mathcal{L} \cap \bigcup_{j \in \mathcal{T}'} E_j \geq m$. Since the vertical line $x = t$ intersects $V$ in $m$ points, this implies that $F$ is $(1, m)$-Furstenberg.

Next we bound the size of $F$. Because $t \in \mathcal{T}'$, for $j \in S, k \in T$, $k + \frac{t+1}{t} j$ can be written as $\sum_{i=t+1}^{l-1} c_i w^i$ for $c_i \in \mathbb{F}_p$. Therefore $|E_j| \leq p^{l-t-1}$. It follows that

$$|F| \leq m + p^{l-t-1} m,$$

completing the proof. □

When $\frac{q}{p^{2t+2}} \leq m \leq \frac{q}{p^{2t+1}}$, it is easy to see that Theorem 4.2 is stronger than inequality (1). When $\frac{q}{p^{2t+3}} \leq m \leq \frac{q}{p^{2t+2}}$, Theorem 4.2 is within a factor of $\sqrt{p}$ of inequality (1). Theorems 4.1 and 4.2 therefore constitute strong upper bounds on $K(q, n, k, m)$ for non-prime $q$.

We also obtain the following corollary, which implies that Theorem 4.2 is optimal up to a constant for $m = \frac{q}{p}$.

**Corollary 1.** *Let $p$ be a prime, $q = p^l$, and $m = \frac{q}{p}$. Then $\frac{1}{2} m^2 \leq K(q, 2, 1, m) \leq m^2 + m$.*

*Proof.* Let $F$ be the Furstenberg set and take $m$ lines which intersect $F$ in at least $m$ points each. This yields at least $m^2 - \binom{m}{2}$ as any two lines intersect in at most 2 points. Therefore $K(q,2,1,m) \geq \frac{1}{2}m^2$.

The upper bound is given by letting $t = 0$ in Theorem 4.2, as $m = p^{l-1}$ hence we obtain $K(q,2,1,m) \leq m^2 + m$. □

The improvements made in Theorem 4.2 to the expected upper bound (1) suggest that the value of $K(q,2,1,m)$ is dependent on the existence of subfields of $\mathbb{F}_q$. This is because the proof heavily relies upon the field extension, and we are not aware of any known constructions of $(1,m)$-Furstenberg sets in prime fields with size matching known lower bounds, except when $m = 0$, $m = 1$, or $m = Cq$ for some constant $C$.

Analogues of Theorems 4.1 and 4.2 also apply to Furstenberg sets in $\mathbb{F}_q^n$ for $n \geq 3$. It suffices to take the idea of the proofs of Theorems 4.1 and 4.2 and extend the construction $E_t = \{(t, j + (j+k)t)\}$ to more coordinates.

Furthermore it is easy to see that the restriction to prime $p$ in Theorem 4.2 is not necessary. These generalizations are summarized in the theorem below.

**Theorem 4.3.** *Suppose $q = p^l$, where $p$ is a prime power and $l \geq 2$ is a positive integer. Then*

*(i) If $q \geq m \geq \frac{q}{p}$, $K(q,n,1,m) \leq C_n q^{\frac{n-1}{2}} m^{\frac{n+1}{2}}$.*

*(ii) If $t \in \mathbb{Z}_{\geq 0}$ and $\frac{q}{p^{2t+3}} \leq m \leq \frac{q}{p^{2t+1}}$, then $K(q,n,1,m) \leq C_n p^{(n-1)(l-t-1)} m$.*

*In both parts, $C_n$ is a constant depending only on $n$.*

*Proof.* (i) We combine the coordinate extension idea with a simple induction on $n$. The case of $n = 2$ is given by Theorem 4.1. Now suppose the statement holds for Furstenberg sets in $\mathbb{F}_q^{n-1}$; we will prove that it holds for sets in $\mathbb{F}_q^n$.

Let $S, T, T', \mathcal{T}, \mathcal{T}'$ be the same as in the proof of 4.1. Let $F_{n-1}$ be the smallest $(1, m)$-Furstenberg set in $\mathbb{F}_q^{n-1}$. For every point $(a_1, \ldots, a_{n-1}) \in F_{n-1}$, append a 0 in the first coordinate, yielding $(0, a_1, \ldots, a_{n-1})$. Let this new set be $F'_{n-1}$.

Then let

$$E_t = \{(t, j_1 + (j_1 + k_1)t, j_2 + (j_2 + k_2)t, \ldots, j_{n-1} + (j_{n-1} + k_{n-1})t)\},$$

where each $j_i$ ranges in the set $S'$ and each $k_i$ ranges in the set $T'$. We claim that the set $F_n = F'_{n-1} \cup \bigcup_{t \in \mathcal{T}'} E_t$ is $(1, m)$-Furstenberg.

Indeed, for any direction vector $v$ with nonzero first coordinate, we may assume by scaling that it has first coordinate 1. Then we may write it as $(1, j_1 + k_1, \ldots, j_{n-1} + k_{n-1})$ for some $j_i \in S$, $k_i \in T$, and hence there are at least $m$ points, one for each $t \in \mathcal{T}'$, on the line $\{(t, j_1 + (j_1 + k_1)t, \ldots, j_{n-1} + (j_{n-1} + k_{n-1})t)\}_t$.

Furthermore, for each direction $u$ with first coordinate 0, by the definition of $F_{n-1}$ there is some line $L$ parallel to $u$ with $|L \cap F_{n-1}| \geq m \implies |L \cap F_n| \geq m$.

Using the inductive hypothesis and the same analysis as the proof of 4.1, we obtain $|F_n| \leq C_n q^{\frac{n-1}{2}} m^{\frac{n+1}{2}}$ where $C_n$ depends only on $n$.

(ii) The proof is in the same vein as that of (i). We apply induction on $n$, with the base case $n = 2$ being Theorem 4.2. Now suppose the statement holds for Furstenberg sets in $\mathbb{F}_q^{n-1}$. We will prove it for sets in $\mathbb{F}_q^n$.

Define $S, T, \mathcal{T}, \mathcal{T}'$ in the same way as the proof of 4.2. Let $F_{n-1}$ be the smallest $(1, m)$-Furstenberg set in $F_q^{n-1}$. For every point $(a_1, \ldots, a_{n-1}) \in F_{n-1}$, append a 0 in the first coordinate, yielding $(0, a_1, \ldots, a_{n-1})$. Let this new set be $F'_{n-1}$. Let

$$E_t = \{(t, j_1 + (j_1 + k_1)t, j_2 + (j_2 + k_2)t, \ldots, j_{n-1} + (j_{n-1} + k_{n-1})t)\},$$

where each $j_i$ ranges in the set $S$ and each $k_i$ ranges in the set $T$. Then by the definitions of $S, T$ and the exact same argument as in the proof of (i), the set $F_n = F'_{n-1} \cup \bigcup_{t \in \mathcal{T}'} E_t$ is $(1, m)$-Furstenberg.

Furthermore, for the same reasons as the proof of Theorem 4.2, $|E_j| \leq p^{(l-t-1)(n-1)}$.

Applying the inductive hypothesis it follows that $|F_n| \leq C_n p^{(l-t-1)(n-1)} m$. $\square$

We end this section by noting that the Ellenberg-Erman bound [5] implies that Theorem 4.3 is optimal up to a constant when $m = \frac{q}{p}$. This therefore generalizes Corollary 1.

# 5 Lower Bounds for Furstenberg Set Sizes

In addition to constructive upper bounds, lower bounding Furstenberg set sizes is also of great interest. As remarked in Section 1, Theorem 3.2 constitutes an improvement of the Ellenberg-Erman bound $Cm^{\frac{n}{2}}$.

At least in the case $n = 2$, the restriction to prime $q$ is not completely necessary. Directly following a remark of Zhang [6] and the methods of [7] we may obtain a similar result for most finite fields.

**Theorem 5.1.** *Let $l$ be an odd positive integer and $q = p^l$ where $p$ is a prime. Then there exist constants $\delta, C > 0$ depending only on $l$ such that $K(q, 2, 1, \sqrt{q}) > Cq^{1+\delta}$.*

*Proof.* The main ingredient is the following Szemeredi-Trotter bound in $\mathbb{F}_q$ noted by Bourgain, Katz, and Tao [7].

**Lemma 5.2.** *Let $p$ be a prime and $q = p^l$ for some odd integer $l$. Let $N \gg 1$ be an integer and $P \subset \mathbb{F}_q^2$, $L$ be a point set and line set, respectively, such that $q^{\frac{1}{4}} \leq |P|, |L| \leq N$. Then there is an absolute constant $C$ and constants $\frac{C}{l} > \epsilon > 2\delta > 0$ depending only on $l$ such that if $q^{1-\delta} \leq N \leq q^{1+\delta}$, then the number of point-line incidences satisfies*

$$|\{(p, l); p \in P, l \in L\}| \leq N^{\frac{3}{2}-\epsilon}.$$

The existence of this type of lemma for $l = 1$ was proved in [7] and they noted the way to generalize it. We provide the details here.

*Proof.* Suppose otherwise. Following the proof of Theorem 6.2 by Bourgain, Katz, and Tao in [7] until the use of their sum-product estimate, we obtain the existence of a subset $A''$ of

$\mathbb{F}_q$ such that $|A''| \geq N^{\frac{1}{2}-\epsilon'}$, and both $|A''+A''|$ and $|A''\cdot A''|$ are at most $N^{\frac{1}{2}+\epsilon'}$ for $\epsilon' \leq C\epsilon$. Here $C$ is a universal constant. Here, $A''+A''$ is the set of all numbers which can be written as the sum of two elements of $A''$, and $A''\cdot A''$ is the set of all numbers which can be written as the product of two elements of $A''$.

By Theorem 4.3 in [7] there exists some subfield $G$ of $\mathbb{F}_q$ with $\frac{1}{2}N^{\frac{1}{2}-\epsilon''} \leq |G| \leq N^{\frac{1}{2}+2\epsilon''}$, where $\epsilon'' \leq C'\epsilon'$ for a universal constant $C'$, as long as $\epsilon$ is sufficiently small. Because $G \subset \mathbb{F}_q$ is a field we know $|G| = p^j$, $j \in \mathbb{Z}$. But

$$p^{l[\frac{1}{2}-\epsilon''][1-\delta]} \leq |G| \leq p^{l[\frac{1}{2}+2\epsilon''][1+\delta]}.$$

Therefore, by choosing $\epsilon$ and $2\delta < \epsilon < \frac{C}{l}$ where $C$ is a sufficiently small absolute constant, $|G|$ cannot be an integer power $p^j$ of a prime because $l$ is odd. This contradiction completes the proof. □

Now let $\delta$, $\epsilon$, and $C$ to be the same as in Lemma 5.2. If $|P| \leq q^{1+\delta}$ we are immediately done. Otherwise $|P| \geq q$ so we may set $N = |P|$. By Lemma 5.2,

$$q\sqrt{q} = qm \leq |\{(p,l), p \in P, l \in L\}| \leq |P|^{\frac{3}{2}-\epsilon}$$

and therefore $|P| \geq q^{1+\frac{1}{2}\epsilon} \geq q^{1+\delta}$.

□

Note that the proof only applies for odd $l$ because Lemma 5.2 only holds in this case. In fact, as remarked in Section 4, if $l$ is even the Ellenberg-Erman bound is optimal up to a constant and therefore cannot be improved in this way.

# 6 Relations Between Furstenberg Set Sizes

The constructions in 4.1 and 4.2 suggest that Furstenberg sets in $\mathbb{F}_p^{kn}$ and $\mathbb{F}_{p^k}^n$ ought to be related due to the field extension idea. Dvir, Dhar, and Lund [4] prove the bound

$$K(q,kr,k,m) \geq \left(\frac{m}{2}\right)^r \tag{4}$$

via finding such a relation when $k = 1$. In it they give a way to formalize this relation.

Let $q$ be a prime power and suppose $n = rk$ for integers $n, r, k$. Consider the field extension $\mathbb{F}_{q^k} = \mathbb{F}_q(w)$. We can write any element $x \in \mathbb{F}_{q^k}$ as $x_0 + x_1 w + \ldots + x_{k-1} w^{k-1}$. Then we may define the following function taking points in $\mathbb{F}_q^n$ to points in $\mathbb{F}_{q^k}^r$.

**Definition 6.1.** For any point $P = (x_0, x_1, \cdots, x_{n-1}) \in \mathbb{F}_q^n$, define
$\sigma(P) = (x_0 + x_1 w + \cdots + x_{k-1} w^{k-1}, x_k + \cdots + x_{2k-1} w^{k-1}, \cdots, x_{(r-1)k} + \cdots + x_{rk-1} w^{k-1}) \in \mathbb{F}_{q^k}^r$.

It is clear that $\sigma$ is a bijection. Let $\tau$ be the inverse of $\sigma$. We can define $\sigma$ and $\tau$ for point sets as well.

**Definition 6.2.** Let $L$ be a set of points in $\mathbb{F}_{q^k}^r$. Then $\sigma(L) = \{\sigma(P)\}$, where $P$ ranges all elements of $L$.

Similarly, if $L'$ is a set of points in $\mathbb{F}_q^n$, then $\tau(L) = \{\tau P\}$, where $P$ ranges all elements of $L'$.

The following lemma generalizes an observation in [4] for flats of higher dimensions.

**Lemma 6.1.** *Let $L$ be a $t$-flat in $\mathbb{F}_{q^k}^r$. Then $\tau(L)$ is a $tk$-flat in $\mathbb{F}_q^n$. In addition, any translate $\tau(L) + u$, $u \in \mathbb{F}_q^n$, can be expressed as $\tau(L + u')$ for some $u' \in \mathbb{F}_{q^k}^r$.*

*Proof.* We may write $L$ as the set of points generated by some $t$ linearly independent vector $u_1, \ldots, u_t$. Specifically,

$$L = \{a + a_1 u_1 + a_2 u_2 + \cdots + a_t u_t\}$$

for $a, a_i \in \mathbb{F}_{q^k}$ and $u_i \in \mathbb{F}_{q^k}^r$.

By properties of field extensions, for any $N$, $w^N$ can be written as a linear combination of $w^j$ for $j = 0, 1, \ldots, k-1$. Therefore, given fixed $u_i$, write each one coordinate of $u_i$ as a linear combination of $w^j$, $j = 0, 1, 2, \ldots, k-1$. Doing the same for each of the variables $a_i$ yields $tk$ variables $y_1, \ldots, y_{tk}$, $k$ for each $a_i$, each in $\mathbb{F}_q$.

Through multiplying and again using the reduction from $w^N$ to a linear combination of $w^j$, $j = 0, 1, 2, \ldots, k-1$, we see that each element of $L$ is of the form

$$a + (x_0 + x_1 w + \cdots + x_{k-1} w^{k-1}, x_k + \cdots + x_{2k-1} w^{k-1}, \ldots, x_{(r-1)k} + \cdots + x_{rk-1} w^{k-1}),$$

where each of the $x_i$ is a fixed linear combination of the $y_i$.

Because any element of $\tau(L)$ is of the form $a' + (x_0, x_1, \ldots, x_{tk-1})$, we may rewrite it by isolating each of the $y_i$ as $a' + \sum_{i=0}^{tk-1} y_i v_i$ for fixed $v_i \in \mathbb{F}_q^n$. This implies that $\tau(L)$ is completely generated by $tk$ vectors, hence it is a subset of some $tk$-flat in $\mathbb{F}_q^n$.

However, observe that $|\tau(L)| = |L| = (q^k)^t = q^{tk}$, which is the size of any $tk$-flat in $\mathbb{F}_q^n$. Therefore $\tau(L)$ must be exactly some $tk$-flat in $\mathbb{F}_q^n$. This proves the first half of the lemma.

To prove the second half of the lemma, it suffices to observe that the vectors generating $\tau(L + u')$ are the same as those generating $\tau(L)$. Because the shift variable $a$ for $L$ can be chosen arbitrarily and corresponds to every possible shift of $\tau(L)$, we may replace $L$ by $L' = \{a' + a_1 u_1 + \cdots a_t u_t\}$ to yield $\tau(L') = \tau(L) + u$. □

Lemma 6.1 shows connections between different dimensional flats in similar fields. This connection implies the following theorem.

**Theorem 6.2.** *Suppose that $n = rk$ and $t$ are positive integers. If $t \le r$, then $K(q, n, kt, m) \ge K(q^k, r, t, m)$.*

*Proof.* Let $F$ be the smallest $(kt, m)$-Furstenberg set in $\mathbb{F}_q^n$. By definition, $|F| = K(q, n, kt, m)$.

Let $F' = \sigma(F)$. Take a $t$-flat $L$ of $\mathbb{F}_{q^k}^r$. By Lemma 6.1, $\tau(L)$ is a $kt$-flat in $\mathbb{F}_q^n$. Because $F$ is Furstenberg, there is some $u \in \mathbb{F}_q^n$ such that $|(\tau(L)+u) \cap F| \ge m$. Applying Lemma 6.1 again, there is $u' \in \mathbb{F}_{q^k}^r$ such that $|\tau(L+u') \cap F'| \ge m$. Because $\tau$ is a bijection, $|(L+u') \cap F| \ge m$. Because $L$ can be any $t$-flat, this implies that $F'$ is $(t, m)$-Furstenberg.

Therefore $|F'| \ge K(q^k, r, t, m)$. Because $|F'| = |F|$, we conclude that $K(q, n, kt, m) \ge K(q^k, r, t, m)$. □

Like Lemma 6.1, Theorem 6.2 serves as comparison between Furstenberg sets in different fields and confirms the importance of the structure of non-prime fields, in particular the existence of subfields and the field extension idea. In addition, known Furstenberg bounds can be used to generate other bounds, such as inequality (4).

# 7 Structure of Small Furstenberg Sets

In Zhang's proof of Theorem 3.4, he utilizes a construction of a small Furstenberg set which has the structure of being the union of several vertical arithmetic progressions. If such a structure were present in all small Furstenberg sets over prime fields, we would be able to work with these sets more concretely and likely reduce them to sumset estimates and related problems.

However, if we apply any invertible linear transformation to a Furstenberg set, we obtain another Furstenberg set which may not be a union of vertical arithmetic progressions. This yields for each small Furstenberg set a new class of such sets of the same size. Considering the invertible linear transformations of $\mathbb{F}_q^2$ motivates us to define a notion of equivalence for Furstenberg sets via the following group theory lens.

Consider the group $G$ of invertible linear transformations and the set $\mathfrak{F}$ of Furstenberg sets of a fixed size. Because invertible linear transformations preserve incidences and therefore send Furstenberg sets to other Furstenberg sets, we obtain a well-defined group action:

$$\begin{aligned} G \times \mathfrak{F} &\longrightarrow \mathfrak{F} \\ (g, F) &\longmapsto g \cdot F \equiv \{g(p) : p \in F\}. \end{aligned}$$

With this setup, we may now define our equivalence notion.

**Definition 7.1.** Two Furstenberg sets $F_1$ and $F_2$ are *equivalent* and we write $F_1 \sim F_2$ if there is some $g \in G$ such that $g \cdot F_1 = F_2$.

It is easy to see that $\sim$ is an equivalence relation. Now we are ready to state our formal conjecture about the structure of small Furstenberg sets.

**Conjecture 7.1.** There are positive real constants $C_1$ and $C_2$ such that for all large primes $q$, any $(1,m)$-Furstenberg set $F$ in $\mathbb{F}_q^2$ of size at most $C_1 q^{\frac{1}{2}} m^{\frac{3}{2}}$ must be equivalent under $\sim$ to a $(1,m)$-Furstenberg set $F'$ containing a subset of at least $C_2|F|$ points which form the union of several vertical progressions.

Note that the equivalence classes under $\sim$ are precisely the orbits of the elements of $\mathfrak{F}$ under the action of $G$ described above. Therefore, to make progress in Conjecture 7.1, it suffices to show that there are few orbits, or even that the orbit of the Furstenberg set $\mathcal{F}$ constructed by Zhang in his proof of Theorem 3.4 is large. If there is just one orbit under the action, then all small Furstenberg sets are equivalent to $\mathcal{F}$. In general, for any $F \in \mathfrak{F}$, there is a $\frac{|O_F|}{|\mathfrak{F}|}$ probability that any Furstenberg set is equivalent to $F$, where $O_F$ is the orbit containing $F$.

Using the orbit-stabilizer theorem, for any element $F \in \mathfrak{F}$, $|O_F| = \frac{|G|}{|S_F|}$ where $S_F$ is the stabilizer subgroup of $F$. To make progress on 7.1, it would suffice to show that the particular stabilizer $|S_{\mathcal{F}}|$ is small. We now give an example of using this method in the simple case of $(1,1)$-Furstenberg sets over $\mathbb{F}_2^2$.

**Example 7.1.** *Consider the set of* $(1,1)$*-Furstenberg sets over* $\mathbb{F}_2^2$ *of size* $1$*. It is easy to see that the smallest such sets are just the four singleton sets,* $F_{ij} = \{(i,j)\}$*,* $i,j = 0,1$*.*

*By checking all possibilities, the group of invertible linear transformations on* $\mathbb{F}_2^2$ *consists of* $6$ *matrices. In addition, it is easy to verify that* $|S_{F_{ij}}|$ *equals* $2$ *when* $(i,j) = (1,0),(0,1),(1,1)$ *and equals* $6$ *when* $(i,j) = (0,0)$*.*

*Hence the orbit of* $F_{ij}$ *has size* $|O_{F_{ij}}| = 3$ *for* $(i,j) = (1,0),(0,1),(1,1)$ *and* $|O_{F_{ij}}| = 1$ *for* $(0,0)$*, implying that 3 of our Furstenberg sets have the same structure and the fourth is different under linear transformations.*

In general, calculating the size of the stabilizer subgroups is much more difficult. One advantage, however, is that by recasting this aspect of the Furstenberg problem in the language of group theory, we may be able to employ tools in the latter field to resolve it.

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