# Boundaries on the Number of Points in Acute Sets

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## Abstract

A famous Erdős problems asks for the maximum cardinality of the sets of points determining only angles less than or equal to $\frac{\pi}{2}$. We examine similar problem but for sets in $\mathbb{R}^d$ determining only angles less than given angle $\theta$. Our results hold for values of $\theta$ in different intervals. First, we prove that any set of points determining only angles less than or equal to $\frac{\pi}{3}$ has maximal cardinality $d+1$ and it is attained if and only if the polytope defined by all points in the set is a simplex. We also find a lower bound for the maximal cardinality of sets with angles $< \theta$, when $\theta \in (\frac{19\pi}{45}, \frac{\pi}{2})$. Lastly, we prove that every set determining angles less than or equal to $\theta$ is finite for $\theta < \pi$.



## Summary

It is relatively easy to find that there are at most three points in the plane, determining only acute angles. Many decades ago, the famous mathematician Paul Erdős conjectured that if there is a sufficiently large number of points in high-dimensional space then they determine at least one obtuse angle. The problem to give approximate solution by finding boundaries was posed as a *prize problem* by the Dutch Mathematical Society, but solutions were only received for low dimensions. The problem is not trivial even when we work in $3D$.

Our results concern the maximum size of a set with only angles less than a certain angle $\theta$. We prove that a set, determining only angles less than or equal to $60°$, contains at most $d+1$ points and is equal to $d+1$ if and only if the points construct a simplex (a generalization of a triangle in high dimensions). We also find a lower bound for sets with angles less than $\theta$, when $\theta$ is a large enough acute angle and prove that the points are not infinitely many when $\theta$ is a large angle but not a line.

# 1 Introduction

The field of combinatorial geometry offers a variety of problems related to the structure of sets. In 1957, Erdős [1] conjectured that in every set of more than $2^d$ points in the $d$-dimensional Euclidean space at least one of the angles determined by three of the points is obtuse. We know that $2^d$ is a tight upper bound as the vertex set of the $d$-dimensional unit cube $\{0, 1\}^d \subseteq \mathbb{R}$ satisfies the non-obtuse set property and has cardinality $2^d$.

In 1962, Danzer and Grűnbraum [2] gave a proof of the conjecture and posed the following question: what is the maximal number of points $S$ in $\mathbb{R}^d$ such that all angles determined by these points are acute? Sets with such property are called *acute sets* and their maximal cardinality is denoted by $f(d)$.

During the years, the problem has attracted the attention of many iconic mathematicians, which has lead to great progress on the bound improvement.

The 2D case has a simple solution, which asserts that the maximal cardinality is equal to 3. The complexity of the problem grows exponentially with the number of dimensions and is not trivial even for $d = 3$. The 3D case was solved by Croft [3] which gave us the result that $f(3) = 5$.

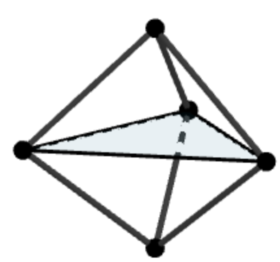


Figure 1: Construction for $f(3) = 5$

Danzer and Grűnbraum also proposed $f(d) \geq 2d - 1$ as a linear lower bound for the maximal cardinality but in 1983 Erdős and Fűredi [4] improved this result to an exponential bound in higher dimensions. By a non-constructive proof and via probabilistic methods, they showed that

$$f(d) \geq \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)^d \approx 0.5 \cdot 1.155^d.$$

They also claimed (without a proof) that more random process than choosing points from the vertex set of the hypercube yields $f(d) \geq 1.189^d$. In 2006 the lower bound was improved by Bevan [5] to

$$f(d) \geq 2\left\lfloor \frac{1}{3}\left(\frac{2}{\sqrt{3}}\right)^{d+1} \right\rfloor \approx 0.77 \cdot 1.155^d.$$

In 2009 a progress on the problem was made by Ackerman and Ben-Zwi [6] when they improved the lower bound to

$$f(d) = \Omega\left(\left(\frac{2}{\sqrt{3}}\right)^d \sqrt{d}\right).$$

In 2011, the lower bound was further improved by Harangi [7]:

$$f(d) = \Omega\left(\left(\sqrt[10]{\frac{144}{23}}\right)^d\right) \approx \Omega(1.201^d).$$

In April 2017, Zakharov [8] presented a new method to attack the problem which includes construction of a new set in $\mathbb{R}^{d+2}$ from one in $\mathbb{R}^d$ implying that $f(d+2) \geq 2f(d)$. The new set is of size $2^{d/2+1}$ which significantly increases the lower bound for $f(d)$ to approximately $1.414^d$.

In July 2017, Cohen [9] refined Zakharov's approach and obtained a new lower bound which is of order approximately $1.43^d$. He also conjectured that $(\sqrt[e]{e}-\epsilon)^d < f(d) < (\sqrt[e]{e}+\epsilon)^d$ are tight bounds on the cardinality. Only few days later, Zakharov [10] significantly improved the bound again to $1.618^d$, thus disproving Cohen's conjecture.

Our research examines sets of points determining angles less than $\theta$ for $\theta \in (0^\circ, \pi)$.

We consider different intervals for the angles and determine when the bounds for their maximal cardinality jumps from linear to exponential. In Section 2 we prove that the maximal

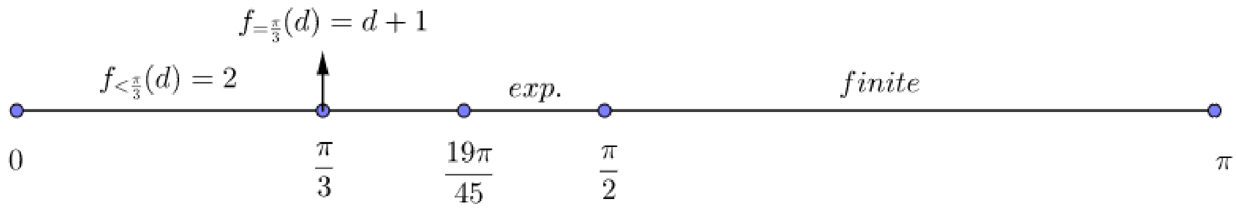



Figure 2: Changes in the values for $f_\theta(d)$

cardinality $f_{\leq\frac{\pi}{3}}(d)$ of a set containing only angles smaller than or equal to $\frac{\pi}{3}$ is at most $d+1$ with equality if and only if the points are vertices of a $d-$dimensional simplex. In Section 3 we give a lower bound for sets with angles less than $\theta$, when $\theta \in (\frac{19\pi}{45}, \frac{\pi}{2})$. Our bound is exponential, and is equal to

$$2\left\lfloor \frac{\sqrt{2}}{3\sqrt{3}} \left( \left( \frac{1}{4\cos\theta} \right)^{\cos\theta} \left( \frac{3}{4(1-\cos\theta)} \right)^{1-\cos\theta} \right)^d \right\rfloor .$$

In section 4, we prove that the sets in which all angles are $\leq \theta$, when $\theta \in (\frac{\pi}{2}, \pi)$, have finite number of points.

## 2 Upper bound for sets of points determining angles $\leq \frac{\pi}{3}$

In this section we use an induction to derive an upper bound for sets of points which determine only angles less than or equal to $\frac{\pi}{3}$.

**Theorem 1.** *For every $d \geq 2$, if a set of points $S \subseteq \mathbb{R}^d$ determines only angles $\leq \frac{\pi}{3}$, then its cardinality is at most $d+1$ and it is exactly $d+1$ if and only if the points define a simplex.*

*Proof.* The proof is based on induction on the number of dimensions. The base case is $d = 2$.

Suppose there is a set of 4 points in the plane which form only angles less than or equal to $\frac{\pi}{3}$. Since they necessarily form a quadrilateral, there is at least one internal angle larger than $\frac{\pi}{3}$ - contradiction. Thus, the cardinality $f_{\leq\frac{\pi}{3}}(2)$ is at most 3. The only sets of 3 points which determine only angles of this size is an equilateral triangle (2-dimensional simplex).

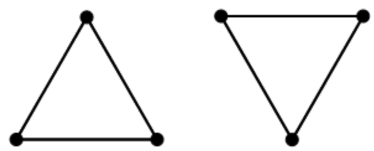


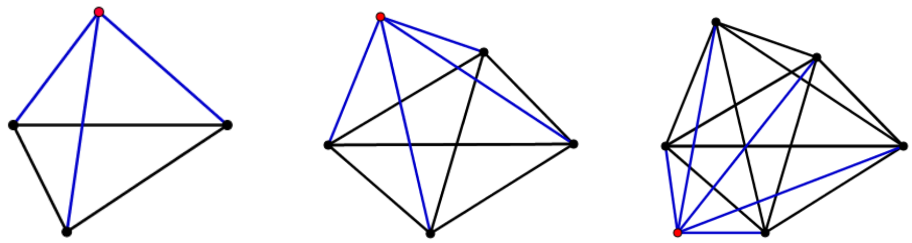
Figure 3: Simplices in $\mathbb{R}^2$

Assuming that the induction hypothesis $f_{\leq\frac{\pi}{3}}(n) \leq n+1$ is true, we need to prove that $f_{\leq\frac{\pi}{3}}(n+1) \leq n+2$ is also true.

Let us suppose that there exists a set with more than $n+2$ points in $\mathbb{R}^{n+1}$ which contains only angles less than or equal to $\frac{\pi}{3}$. Therefore, there exist $n+3$ points $v_1, v_2, ..., v_{n+3}$ in $\mathbb{R}^{n+1}$, such that all determined angles are less than or equal to $\frac{\pi}{3}$. By induction hypothesis we may fix a basis , so that the vectors $v_1, v_2, ..., v_{n+3}$ lie in a hyperplane, defined by $x_1 + x_2 + \cdots + x_{n+2} = 1$, in which their coordinates are given by

$$v_1 = (1, 0, 0, ..., 0)$$
$$v_2 = (0, 1, 0, ..., 0)$$
$$\vdots$$
$$v_{n+1} = (0, 0, 0, ..., 1, 0).$$

We denote the coordinates of $v_{n+2}$ by $(y_1, y_2, ..., y_{n+2})$. Because every angle is less than or equal to $\frac{\pi}{3}$, every 3 points in the set $S$ must form an equilateral triangle. Our induction hypothesis and construction lets us calculate the length $|\overrightarrow{v_1v_2}|$, which is equal to $\sqrt{2}$; hence $|\overrightarrow{v_iv_j}| = \sqrt{2}$ for any $i \neq j$, with $i, j \in \{1, 2, ..., n+3\}$.



(a) Simplex in $\mathbb{R}^3$ (b) Simplex in $\mathbb{R}^4$ (c) Simplex in $\mathbb{R}^5$

Figure 4: We mark $v_{n+2}$ in red and the new-formed sides in blue. The length of the sides is $\sqrt{2}$.

To find the angle between $(v_{n+2} - v_1)$ and $(v_{n+3} - v_1)$, we solve the following system of equations, which determines the coordinates of $v_{n+2}$ and $v_{n+3}$.

$$
\begin{aligned}
|\overrightarrow{v_1 v_{n+2}}| &= \sqrt{(y_1-1)^2+(y_2-0)^2+(y_3-0)^2+\cdots+(y_{n+1}-0)^2+(y_{n+2}-0)^2} = \sqrt{2}\\
|\overrightarrow{v_2 v_{n+2}}| &= \sqrt{(y_1-0)^2+(y_2-1)^2+(y_3-0)^2+\cdots+(y_{n+1}-0)^2+(y_{n+2}-0)^2} = \sqrt{2}\\
|\overrightarrow{v_3 v_{n+2}}| &= \sqrt{(y_1-0)^2+(y_2-0)^2+(y_3-1)^2+\cdots+(y_{n+1}-0)^2+(y_{n+2}-0)^2} = \sqrt{2}\\
&\vdots\\
|\overrightarrow{v_{n+1} v_{n+2}}| &= \sqrt{(y_1-0)^2+(y_2-0)^2+(y_3-0)^2+\cdots+(y_{n+1}-1)^2+(y_{n+2}-0)^2} = \sqrt{2}.
\end{aligned}
$$

Hence we have equality for the first $y+1$ coordinates

$$y_1 = y_2 = y_3 = \cdots = y_{n+1}.$$

Then we substitute $y_i$ with $y_1$ for $i \in \{2, 3, \cdots, n+1\}$ and obtain $(n+1)^2 y_1^2 + y_{n+2}^2 = 2$. Using the definition of a hyperplane, we also have that $(n+1)a + b = 1$ which implies that the only possible coordinates of the points $v_{n+2}$ and $v_{n+3}$ are

$$v_{n+2} = (0, 0, 0, ..., 0, 1)$$

$$v_{n+3} = \left( \frac{2}{n+2}, \frac{2}{n+2}, \frac{2}{n+2}, ..., \frac{2}{n+2}, -1 \right).$$

Using the values for the coordinates, we calculate the dot product, which is $-1$. It follows that the angle determined by the three points with apex $v_1$ is bigger than $\frac{\pi}{3}$ and in fact is equal to $\pi$, which contradicts our assumption. The above argument shows that there are at most $n+2$ vectors that determine angles less than or equal to $\frac{\pi}{3}$.

The last part is to prove that the maximum of $n+2$ points is achieved only for points defining a simplex.

Without loss of generality, we assume that the points $v_1, v_2, ..., v_{n+2}$ lie in the hyperplane of $\mathbb{R}^{n+2}$, defined by $x_1 + x_2 + \cdots + x_{n+2} = 1$. By induction hypothesis, we also assume that

$$\begin{aligned} v_1 &= (1, 0, 0, ..., 0, 0) \\ v_2 &= (0, 1, 0, ..., 0, 0) \\ &\vdots \\ v_{n+1} &= (0, 0, 0, ..., 1, 0). \end{aligned}$$

The next step in proving the hypothesis is to find the coordinates of the point $v_{n+2}$ and determine whether the polytope formed is a simplex.

Using the same techniques as in the first part of the proof, we obtain the possible coordinates of $v_{n+2}$

$$v_{n+2} = (0, 0, 0, ..., 0, 1)$$

$$v_{n+2} = \left( \frac{2}{n+2}, \frac{2}{n+2}, \frac{2}{n+2}, ..., \frac{2}{n+2}, -1 \right).$$

The dot product of the two possible vectors is $-1$, which means that they are symmetrical with respect to the hyperplane in which we can put the rest $n+1$ points. Both points form a $(n+1)$-dimensional simplex.

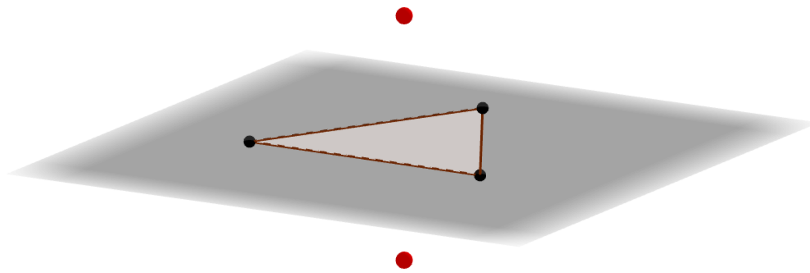


Figure 5: Example of two symmetrical points in $\mathbb{R}^3$

$\square$

## 3 Lower bound for sets of points determining angles $\theta$, when $\theta \in (\frac{19\pi}{45}, \frac{\pi}{2})$

In this section we examine sets of points which determine only angles less than or equal to $\theta$ for $\theta \in (\frac{19\pi}{45}, \frac{\pi}{2})$. In the general case when $\theta = \frac{\pi}{2}$, the main approach used to attack the problem is the probabilistic method. Using the same tool, we consider the vertices of the hypercube, defined as the set $A \subseteq \{0,1\}^d$. We choose a set number of vectors and calculate the expectation for the angles larger than or equal to $\theta$. Given that expectation, we can simply subtract the large angles and obtain the final number of points which determine only angles less than $\theta$.

**Theorem 2.** *For every $d \geq 2$ and $\theta \in (\frac{19\pi}{45}, \frac{\pi}{2})$, there is a set $A \subseteq \{0,1\}^d$ of cardinality*

$$2\left\lfloor \frac{\sqrt{2}}{3\sqrt{3}} \left( \left( \frac{1}{4\cos\theta} \right)^{-\cos\theta} \left( \frac{3}{4(1-\cos\theta)} \right)^{\cos\theta - 1} \right)^d \right\rfloor$$

*in $\mathbb{R}^d$ that determines only angles less than $\theta$.*

*Proof.* Set $m := \left\lfloor \frac{\sqrt{2}}{3\sqrt{3}} \left( \left(\frac{1}{4\cos\theta}\right)^{-\cos\theta} \left(\frac{3}{4(1-\cos\theta)}\right)^{\cos\theta - 1} \right)^d \right\rfloor$ and take $3m$ vectors

$$a(1), a(2), ..., a(3m) \in \{0,1\}^d.$$

We choose their coordinates randomly and independently. Because we use the vertex set of the hypercube, the angles determined by these vectors are either right, or acute. To find an upper bound for the maximal cardinality of the set, we find the expected number of angles larger than $\theta$.

Let us pick the vectors $a(i), a(j)$ and $a(k)$ and denote $a(j) - a(i)$ by $v$, and $a(k) - a(i)$ by $w$. If $v = (v_1, v_2, ...v_d)$ and $w = (w_1, w_2, ..., w_d)$, their dot product $v \cdot w$ is then expressed as

$$v \cdot w = v_1w_1 + v_2w_2 + \cdots v_dw_d = \cos\theta |v||w|,$$

where $|v|$ is the length of $|v|$, $|w|$ is the length of $w$ and $\theta = \angle(v, w)$. The maximal length of $|v|$ or $|w|$ is $\sqrt{{1_1}^2 + {1_2}^2 + \cdots + {1_d}^2}$. We substitute this value and express the dot product as

$$v_1w_1 + v_2w_2 + \cdots + v_dw_d \leq d\cos\theta,$$

$$\prod_{m=1}^{d} (a(j)_m - a(i)_m)(a(k)_m - a(i)_m) \leq d\cos\theta.$$

The number of triples of vectors $a(i), a(j)$ and $a(k)$ we check for acuteness is equal to $3\binom{3m}{3}$ as there are $\binom{3m}{3}$ combinations of vectors and three ways to choose an apex. We now proceed to find the expected number of angles larger than $\theta$. To do this we use the Chernoff-Hoeffding theorem, which states

**Theorem 3** (Chernoff-Hoeffding theorem). *Suppose $X_1, ..., X_n$ are independent and identi-*

*cally distributed random variables, taking values in* $\{0,1\}$*. Let* $p = E[X_i]$ *and* $\epsilon > 0$*. Then*

$$Pr\left(\frac{1}{d}\sum_{l=1}^{d} X_l \leq (p-\epsilon)\right) \leq \left(\left(\frac{p}{p-\epsilon}\right)^{p-\epsilon}\left(\frac{1-p}{1-p+\epsilon}\right)^{1-p+\epsilon}\right)^{d}.$$

Let $X_l = (a(j)_l - a(i)_l)(a(k)_l - a(i)_l)$ and $\epsilon = \frac{1}{4} - \cos\theta$. We know that $X_l$ equals 1 when

$$a(j)_l = a(k)_l = 1 \text{ and } a(i)_l = 0$$

or

$$a(j)_l = a(k)_l = 0 \text{ and } a(i)_l = 1.$$

These are two of total eight cases, which means the probability of $X_l = 1$ is $p = \frac{1}{4}$. We apply the Chernoff-Hoeffding theorem with this probability and obtain

$$Pr\left(\frac{1}{d}\sum_{l=1}^{d} X_l \leq \cos\theta\right) \leq \left(\left(\frac{1}{4\cos\theta}\right)^{\cos\theta}\left(\frac{3}{4-4\cos\theta}\right)^{1-\cos\theta}\right)^{d}.$$

We conclude that the expected number of angles which are larger than $\theta$ is at most

$$3\binom{3m}{3}\left(\left(\frac{1}{4\cos\theta}\right)^{\cos\theta}\left(\frac{3}{4-4\cos\theta}\right)^{1-\cos\theta}\right)^{d}.$$

This means that there is a choice of the $3m$ vectors in which the angles larger than $\theta$ are at most $3\binom{3m}{3}\left(\left(\frac{1}{4\cos\theta}\right)^{\cos\theta}\left(\frac{3}{4-4\cos\theta}\right)^{1-\cos\theta}\right)^{d} < 3\frac{(3m)^3}{6}\left(\left(\frac{1}{4\cos\theta}\right)^{\cos\theta}\left(\frac{3}{4-4\cos\theta}\right)^{1-\cos\theta}\right)^{d} < m$. If there are no more than $m$ combinations of three vectors which lead to angles larger than $\theta$, then we can simply remove $m$ of the vectors. The remaining vectors do not determine angles larger than $\theta$. $\square$

# 4 Upper bound for sets of points determining angles $\leq \theta$, when $\theta < \pi$

In this section we prove that if a set of points determines only angles smaller than a defined angle $\theta$, then this set has finite number of points.

**Theorem 4.** *For every $d \geq 2$ and every $\theta < \pi$, if a set of points $S \subseteq \mathbb{R}^d$ determines only angles less than $\pi$, then $S$ is finite.*

*Proof.* To prove this statement we need to consider two cases for the set $S$ - when $S$ is bounded and when it is unbounded.

**Case 1.** *The set of points $S \subseteq \mathbb{R}^d$ is unbounded.*

We pick the point $A \in S$ and define a sphere with radius $r$ and center $A$. Then we divide the sphere to finitely many identical pieces. The set of points in a piece $K$ is denoted by $P_k$. The pieces are formed such that if we consider the points $i$ and $j$, where $i, j \in P$, the angle with apex $A$, determined by $i, j$ and $A$, is smaller than $\frac{180-\theta}{2}$. We denote a region by $R$ and define it as $R := \{t(p - A), p \in P, t > 0\}$. Let $M = R \cap S$. Because $S$ is an unbounded set, we can choose a particular piece $P_k$ such that $M$ is unbounded. Then we consider the points $B \in M$ and $C \in M$ such that the distance between $A$ and $B$ is at least $m$ and the distance between $B$ and $C$ is larger than $m$. We are interested in $\triangle ABC$ and its angles. Because $BC > AB$ we know that $\angle ACB < \angle CAB < \frac{180-\theta}{2}$. This directly implies that $\angle ABC > \theta$, which contradicts our assumption that there are no angles larger than $\theta$.

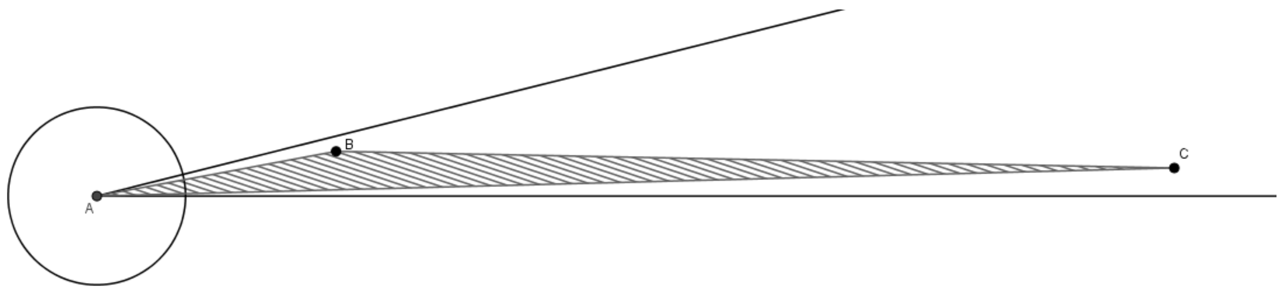



Figure 6: Construction in unbounded $S \subseteq \mathbb{R}^d$

**Case 2.** *The set of points $S \subseteq \mathbb{R}^d$ is bounded.*

We suppose that $S$ is an infinite bounded subset of $\mathbb{R}^d$.

Let us take the sequence $x, x_1, x_2, ..., x_m$, converging to $x$, where $x_i \in S$, $i \in \{1, 2, ..., m\}$. The limit point $x$ is not necessarily in $S$. As in Case 1 we can construct a sphere with center $x$ and divide $\mathbb{R}^d$ into pieces, denoted by $P$. Again, the pieces are formed such that if we consider the points $y$ and $z$, where $y, z \in P$, the angle with apex $A$, determined by $y, z$ and $A$, is smaller than $\frac{180-\theta}{2}$. A region is defined as $R := \{t(p - x), p \in P, t > 0\}$. Because $S$ is infinite set, there must be a region $N$ in which there are infinitely many points from the sequence. Denote the subsequence of $x_1, x_2, ..., x_m$ with $x_j(1), x_j(2), ..., x_j(n)$ and $x_j(i) \in N$. Let $x_j(1) = C$ and take point $B$ to be a point in the subsequence $x_j(i)$, satisfying the following property for the length of the vectors $|C - B| > |B - x|$.

**Theorem 5** (Bolzano-Weierstrass theorem)**.** *Every bounded sequence in $\mathbb{R}^d$ has a convergent subsequence.*

Applying Bolzano - Weierstrass theorem, we may assume that the subsequence

$$x_j(1), x_j(2), x_j(3), \ldots, x_j(n)$$

is converging. Then we take a point $A$ in the subsequence such that the angle determined by $A, B$ and $C$ with apex $A$ is smaller than $\frac{180-\theta}{2}$. This directly implies that the angle $ABC$

is larger than $\theta$.

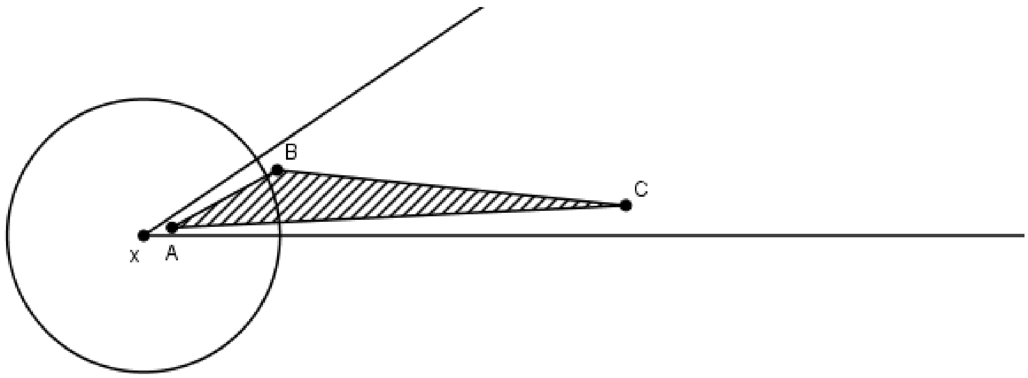



Figure 7: Construction in bounded $S \subseteq \mathbb{R}^d$

$\square$

# 5 Conclusion

We determined that a set determining only angles less than $\frac{\pi}{3}$ is a set of at most $d+1$ points. This bound $d+1$ is attained if and only if the points form a simplex. Via probabilistic methods we give a lower bound on the cardinality of sets with angles only less than $\theta$, when $\theta \in (\frac{19\pi}{45}, \frac{\pi}{2})$. We also prove that the sets in which all angles, determined by the points, are $\leq \theta$, when $\theta \in (\frac{\pi}{2}, \pi)$, have finite number of points.

In the future, we plan continue our examination on the rest of the intervals of angles, for which there are not known bounds. To be more specific, we intend to find an exponential lower bound for the number of points determining only angles $\theta$, when $\theta \in (\frac{\pi}{3}, \frac{19\pi}{45})$ and we aim to give an exponential upper bound for sets with angles $\theta$, when $\theta \in (\frac{\pi}{2}, \pi)$.

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