Characterizing x-Monotone and Outerplanar Thrackles

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Under the Direction of Aaron Potechin

This paper is dedicated to the memory of my grandfather, Jack Salz, who played a huge part in inspiring my interest in research, mathematics, and critical thinking of all kinds.

Abstract: Characterizing x-Monotone and Outerplanar Thrackles

A *graph drawing* is a visual representation of a graph in which points represent vertices and continuous curves represent edges. A *thrackle* is defined as a graph drawing in the plane in which every edge intersects every other edge exactly once. In this study, we focus on two special cases of thrackles: (i) *outerplanar thrackles*, in which the vertices lie on a circle and the edges are contained inside the circle, and (ii) *x-monotone thrackles*, in which the vertices may be anywhere, but each edge may intersect any vertical line in the plane at most once. For both of these cases, we determine the set of graphs that can be drawn in the specified manner. These results may be a step towards solving the general problem of characterizing all thrackles, whose solution would in turn likely lead to advances in many different and widely applicable problems in graph drawings.

Executive Summary: Characterizing x-Monotone and Outerplanar Thrackles

Networks are all around us in the modern world. We are connected physically to others by a network of roads which allows us to quickly travel to almost anywhere on one landmass, and we are connected virtually to almost everyone in the world via networks such as the internet and telephone networks. There are also many examples of networks which are less obvious to the untrained eye. For example, the pattern of transistors on a computer chip can be represented as a network, and properties of the network can be used to optimize the design of the chip. Although these networks are largely of recent origins, mathematicians have been studying networks for several centuries, albeit under a different name. In math, a *graph* is the abstract representation of a network, and a *graph drawing* is a visual representation of a network. In a graph drawing, we draw points to represent the members of the network— for example, everyone who has a Facebook account— and we draw curves between pairs of members which are directly connected— for example, people who are friends on Facebook.

There are many properties of graph drawings which are interesting to study, but one of the most important ones is the way in which the curves in a drawing cross each other. In this paper, we consider an extreme scenario in the spectrum of possible crossing patterns. Specifially, we investigate graph drawings in which each curve in the drawing crosses every other curve; these drawings are called *thrackles*. Given a network, we would like to be able to determine whether the network can be drawn as a thrackle. We focus on two special cases of the problem in which we impose further constraints on the drawing of the graph to make the problem easier to handle. For both of the cases we consider, we determine exactly which networks can be drawn as a thrackle with the specified constraints. The results obtained here may be a step towards solving the general problem of characterizing all thrackles, whose solution would in turn likely lead to advances in many different and widely applicable problems in graph drawings.

1 Introduction

A thrackle is a drawing of a graph in the plane such that every edge intersects every other edge exactly once. A classic example of a thrackle is the six cycle, shown in Figure 1.

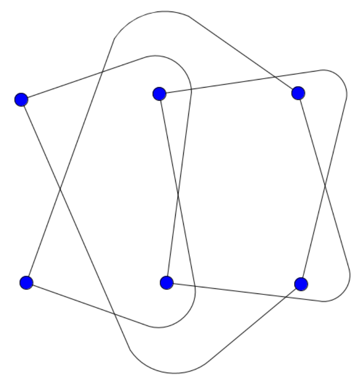

Figure 1: A thrackle drawing of C_6

Over 40 years ago, John H. Conway conjectured that there are at most as many edges as vertices in any graph which can be drawn as a thrackle. At the time, Conway offered a bottle of beer for a solution, and although the prize has since risen to \$1000, the conjecture is still unresolved [3]. There has, nonetheless, been a good deal of research on thrackles.

For example, D. R. Woodall [2] showed that if Conway's conjecture is true, a graph is thrackleable if and only if it contains at most one odd cycle, at most one cycle in each connected component, and no cycle of length four. Woodall also showed that if Conway's conjecture is false, then there is a minimal counterexample consisting of two even cycles sharing one vertex. Thus, to prove the conjecture, it must only be shown that any such graph cannot be drawn as a thrackle. Additionally, several upper bounds on the number of edges in a thrackle have been proven since the proposal of Conway's conjecture. Currently, the tightest proven bound is $\frac{167}{116}v$, where v is the number of vertices, by Fulek and Pach [3].

There have also been several papers examining Conway's conjecture for special cases. Cairns and Nikolayevsky [4] proved Conway's thrackle conjecture for outerplanar thrackles, which are thrackles whose vertices lie on some circle and whose edges lie in the interior of the circle, as shown in Figure 2. Pach and Sterling [5] proved Conway's thrackle conjecture for x-monotone thrackles, which are thrackles whose edges cross each vertical line in the plane at most once, as shown in Figure 3. In this paper, we extend the work of Cairns, Nikolayevsky, Pach, and Sterling by giving a complete characterization of which graphs can be drawn as outerplanar thrackles and which graphs can be drawn as x-monotone thrackles. We hope that the complete solution of the thrackle problem for these special cases will help to give insight into the solution of the general thrackle problem.

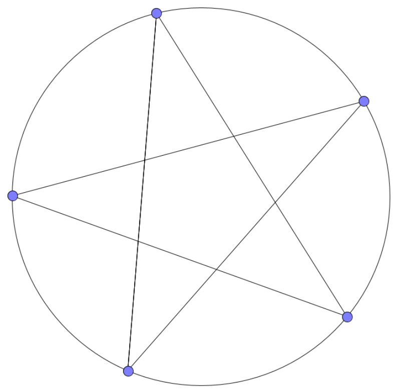

Figure 2: An outerplanar thrackle

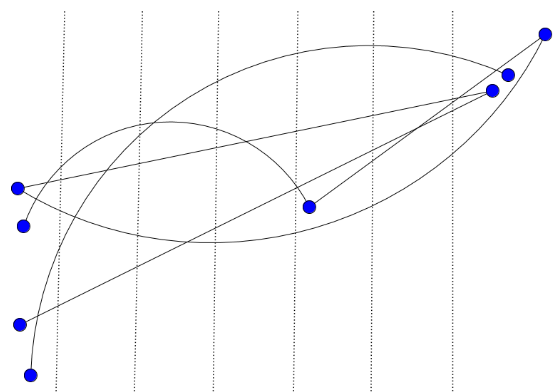

Figure 3: An x-monotone thrackle. The vertical lines are simply visual aids.

In addition to its inherent interest, the thrackle problem is studied because its solution will likely give insight into how to solve a large number of related problems in graph drawings. This field has a variety of applications including bioinformatics, cartography, and social network analysis, but its most significant practical use is probably in very large scale integration (VLSI). As computers have grown in complexity over the past few decades, the efficient and effective layout of the transistors on a computer chip has become increasingly important for their design and progress. In particular, it is desirable to lay out the transistors on a computer chip such that they take up the smallest area possible while avoiding certain types of crossing components [6]. Although there are many algorithms for doing so today, there is a great gap of theoretical knowledge about the problem and there may be a potential for great improvement which would lead to more effective and smaller computer chips.

Now, it has already been shown that there is a close relationship between minimizing the number of crossings in a graph and circuit layout design [6]. While it seems that this problem is the opposite of the thrackle problem, in which we want to create as many intersections as possible, there are actually many connections between the two. For one, maximization and minimization problems in mathematics are often closely related, and in many cases, the same method used to maximize a quantity can be used to minimize it with very little modification. Indeed, there is already concrete evidence of connections between thrackles, which involve crossing maximization, and planar graphs, which involve crossing minimization. One great example of this is a beautiful theorem by Lovsz, Pach, and Szegedy, which states that every thrackleable bipartite graph is also a planar graph [1]. This suggests deep connections between these two edge-crossing extremes and gives more weight to the supposition that studying the thrackle problem could lead to advances in other areas of graph drawings.

In a more general sense, because the thrackle problem is one of the simplest problems one could propose in the field of graph drawings, it is reasonable to believe that the methods developed in finding its solution could lead to a solution to some of the deeper and more directly applicable problems in the field, such as those relating to VLSI. This idea is especially compelling because, beyond certain results about planar graphs, very little progress has been made towards directly solving the fundamental problems of graph drawings. For example, the minimum weight drawability problem for triangulated graphs, the 3D orthogonal representation problem, and the visibility graph recognition problem are all unsolved problems in graph drawings, and even the computational complexities of solving specific cases of the problems are not known [7]. Hopefully, by working on simpler problems such as the thrackle problem, we can work towards an eventual solution to these problems and many other unsolved problems in graph drawings.

In Section 3, we define some basic concepts needed for the rest of the paper. In section 4, we characterize the graphs that can be drawn as outerplanar thrackles. Finally, in section 5, we characterize the graphs that can be drawn as x-monotone thrackles.

2 Basic Definitions

Definition 2.1. *A* graph *is a set V of* vertices *combined with a set E of pairs of vertices, which we call* edges*.*

Definition 2.2. *We say that an edge* connects *the two vertices in its pair, and that it is* incident *to either of the vertices in its pair. We will denote the edge connecting the vertices v and u by either $e_{u,v}$ or (u, v). A vertex is* adjacent *to another vertex, or is the* neighbor *of another vertex, if there is an edge which connects them.*

Definition 2.3. *The* degree *of a vertex is the number of edges which are incident to it.*

Definition 2.4. *A* path *of length n is a sequence of distinct vertices $v_1, v_2, \ldots, v_n$ such that, for all $1 \leq i < n$, v_i is adjacent to v_{i+1}, and a* cycle *is a path which also contains the edge (v_n, v_1).*

Definition 2.5. *A* bipartite graph *is a graph whose vertices can be partitioned into two sets, A and B, such that no two vertices in A are connected and no two vertices in B are connected.*

Definition 2.6. *A* connected component *of a graph G is a set containing a vertex v of G and all other vertices u of G for which there is a path beginning at v and ending at u.*

Definition 2.7. *A* graph drawing *of a graph G is a mapping which takes vertices of G to points in the plane and edges of G to continuous curves in the plane such that the arc corresponding to a given edge begins at one of the vertices it is incident to and ends at the other.*

We also require that no curve intersects itself, that no curve contains any of the vertices other than the two at its endpoints, and that no two of the curves are tangent to each other at any point. Henceforth, we will not distinguish between the points and curves of a graph drawing and the edges and vertices of the underlying graph.

Definition 2.8. *A graph drawing is a* thrackle *if and only if every edge intersects every other edge at exactly one point. A graph is* thrackleable *if it can be drawn as a thrackle.*

3 Outerplanar Thrackles

Definition 3.1. *An* outerplanar thrackle *is a thrackle with all of its vertices on a circle and all of its edges in the interior of the circle.*

To state precisely which graphs can be drawn as outerplanar thrackles, we need a few more definitions.

Definition 3.2. *An* open caterpillar *is a graph containing some path such that every vertex of degree greater than one is in the path and every vertex of degree one is either in the path or adjacent to a vertex in the path.*

Definition 3.3. *We define a* closed caterpillar *in the same way as an open caterpillar, except that the path is replaced by a cycle.*

These two types of graphs are shown in Figure 4.

Now, we characterize all outerplanar thrackles with the following three theorems.

Theorem 3.1. *An outerplanar thrackle contains no even cycle.*

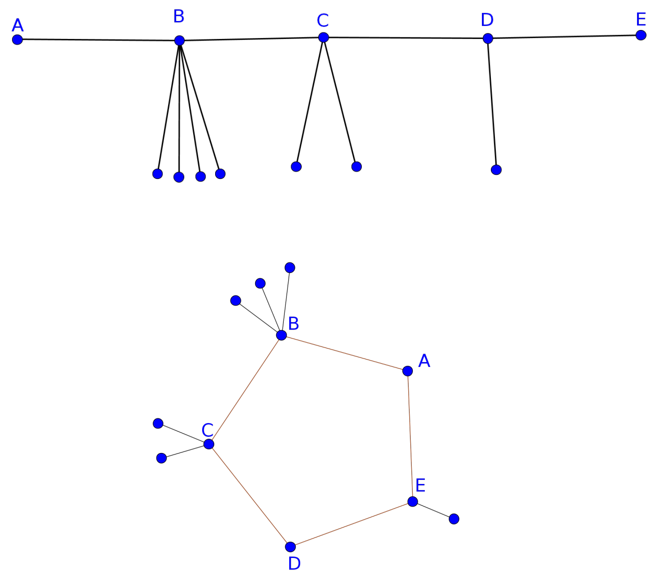

Figure 4: An open caterpillar and a closed caterpillar, respectively

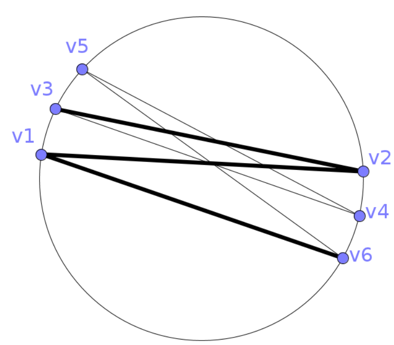

Figure 5: An Attempt to Draw an Even Cycle as an Outerplanar Thrackle

Proof. Assume for the sake of contradiction that there is an outerplanar thrackle with an even cycle, and call the vertices of the cycle $v_1, v_2, \ldots v_n$. Now, for $3 \leq i \leq n$, each edge (v_i, v_{i+1}) must cross the edge (v_1, v_2). Thus, v_i and v_{i+1} are on opposite sides of that edge, and any v_i and v_j with i and j of different parities must be on different sides of (v_1, v_2). This means that, since 3 and n are of opposite parities, v_3 and v_n are on opposite sides of (v_1, v_2). But then, as in Figure 5, (v_1, v_n) and (v_2, v_3) cannot cross, which is a contradiction. $\square$

Theorem 3.2. *If an outerplanar thrackle contains an odd cycle, it is a closed caterpillar.*

Proof. Assume for the sake of contradiction that there is an edge e which is disjoint from the cycle. It is well known that the regions formed by any closed curve in the plane can be two-colored, as shown in Figure 6. Since the cycle must be contained within the circle and its boundary, all points on the boundary of the circle, excepting the vertices of the cycle, lie in the same region of this two-coloring. Now, we know that e must cross every edge in the cycle exactly once between its two endpoints. Since the cycle is odd, this means that the color of the region it is in changes an odd number of times from start to finish, so that it must end in a region which is colored differently than the region it started in. As we already know that its starting and ending point are in the same region, this is a contradiction. Therefore, if the graph contains a cycle, every edge in the graph must be incident to some vertex of the cycle. Now, if we can show that each vertex

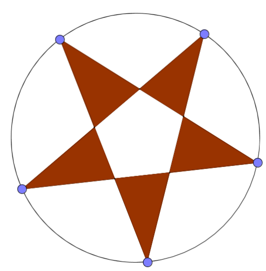

Figure 6: A two-coloring of a 5-cycle

outside the cycle can be adjacent to at most one vertex in the cycle, the proof will be complete. To do this, we use the fact that outerplanar thrackles have at most as many edges as vertices, as proven by Cairns and Nikolayevsky [4]. If a vertex is adjacent to two vertices in the odd cycle, the subgraph containing only the vertex and the cycle has more edges than vertices, which is a contradiction. Thus, each vertex outside the cycle is adjacent to at most one vertex in the cycle, and the graph is a closed caterpillar. □

Theorem 3.3. *If an outerplanar thrackle contains no cycle, every connected component is an open caterpillar.*

Proof. First, we show that every vertex in the graph has at most two neighbors of degree greater than one. Assume for the sake of contradiction that an outerplanar thrackleable graph contains a vertex v with three neighbors a, b, and c, each of degree at least two. Without loss of generality, let the points lie on the circle in the clockwise order v, a, b, c. Now, consider any neighbor u of b which is not v. Without loss of generality, u lies on the same side of segment $e_{v,b}$ as a. However, this means that u and b are on the same side of $e_{v,b}$, so $e_{u,b}$ does not intersect $e_{v,c}$, as shown in Figure 7. This contradicts the assumption that the graph is a thrackle.

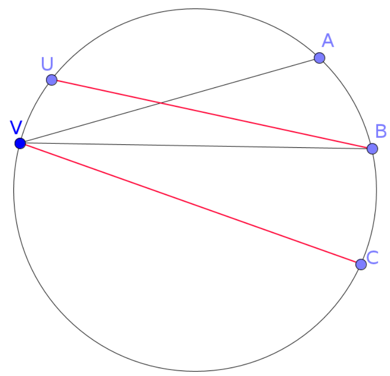

Figure 7: A vertex with three neighbors of degree at least two

Now, consider any connected component C of the graph. Let P be the longest path in C and let v be any vertex which is in C, but not P. Let the length of the shortest path S from v to a vertex u in P be n.

Assume for the sake of contradiction that $n > 1$. If u is on the end or adjacent to the end of P, the path from v to u to the other end of P is longer than P, which is a contradiction. Otherwise, u has two neighbors in P which have degree at least two and one neighbor in S which has degree at least two, so it has a total of three such neighbors, which is also a contradiction. Thus, $n = 1$. Additionally, no vertex in C but not in P may contain two edges connecting it to P, because that would create a cycle in the graph. Thus, every vertex in C but not in P is a degree one vertex adjacent to a vertex in P, and C must be an open caterpillar. $\square$

Now, we must show that we can construct any outerplanar thrackle which meets the specifications of Theorems 3.2 and 3.3. We do this in the following two theorems.

Theorem 3.4. *Every closed caterpillar is outerplanar thrackleable.*

Proof. To construct a closed caterpillar with an odd cycle of length n as an outerplanar thrackle, we begin by drawing an n-pointed star with its vertices equally spaced around the circle. Then, to add a vertex v which is adjacent to the vertex u in the cycle, we place v in between the two neighbors of u, as shown in Figure 8. This completes the construction.

$\square$

Theorem 3.5. *A graph whose every component is an open caterpillar is outerplanar thrackleable.*

Proof. To construct an open caterpillar, we use the same construction as in Theorem 3.4 but do not complete the star. This makes the graph bipartite, so we may flatten out the graph as shown in Figure 9. If it is bipartite under the partition of the vertices into the sets A and B, we place all vertices of A in a small region

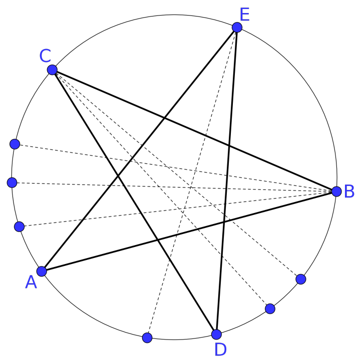

Figure 8: A closed caterpillar drawn as an outerplanar thrackle

a on one side of the circle and all of the vertices of B in a small region b on the other side of the circle. To combine n open caterpillars with regions $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$, we place the regions in the order $a_1 a_2, \ldots, a_n b_1 b_2, \ldots, b_n$ and every edge in every component crosses every edge in every other component, as shown in Figure 10. This completes the proof.

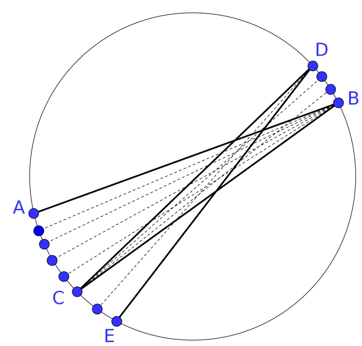

Figure 9: One open caterpillar drawn as an outerplanar thrackle

□

4 x-Monotone Thrackles

Definition 4.1. *An* x-monotone thrackle *is a thrackle in which every edge intersects every vertical line at most once.*

We first define a few useful concepts for working with these thrackles.

Definition 4.2. *A* left vertex *is a vertex which is incident to an edge that approaches the vertex from the right, and a* right *vertex is a vertex which is incident to an edge that approaches the vertex from the left.*

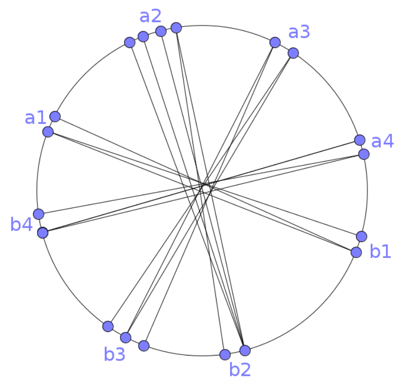

Figure 10: Several open caterpillars drawn as an outerplanar thrackle

Definition 4.3. *A* split point *is a vertex which is both a left vertex and a right vertex.*

Definition 4.4. *A* fork point *is a point which has more than two neighbors of degree at least two.*

Definition 4.5. *A* central neighbor v *of a vertex* u *is a vertex such that at least two edges other than* $e_{u,v}$ *leave* u *in the same horizontal direction as* $e_{u,v}$*, and one of them is above* $e_{u,v}$ *in the vicinity of* u*, and one of them is below* $e_{u,v}$ *in the vicinity of* u*.*

These points are shown in Figure 11.

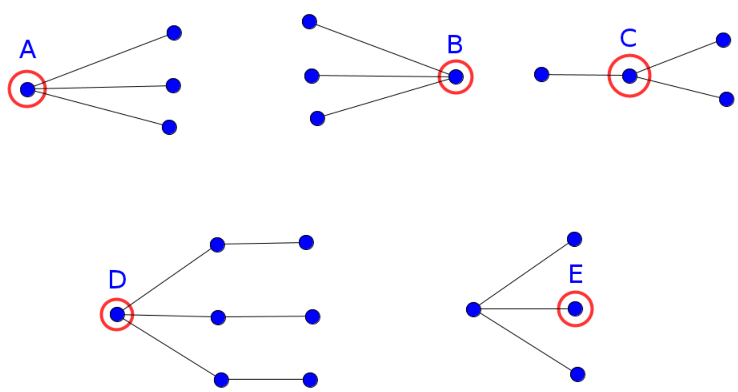

Figure 11: A left vertex, a right vertex, a split point, a fork point, and a central neighbor, respectively

We now prove five lemmas which help characterize x-monotone thrackles.

Lemma 4.1. *There is at most one split point in any* x*-monotone thrackle.*

Proof. Assume that, on the contrary, there are two distinct vertices u and v which are both split points. Without loss of generality, let u be to the left of or on the same vertical line as v. Since u and v are split points, there is an edge which leaves u to the left and one which leaves v to the right. If u is to the left of v, as in Figure 12, these two edges share no x coordinates, so they do not intersect. Thus, u must be on the same vertical line as v. Then, the only shared x coordinate is in line with u and v, but the edges could only intersect there if u was in fact the same point as v. This contradicts the assumption that u and v are distinct vertices. □

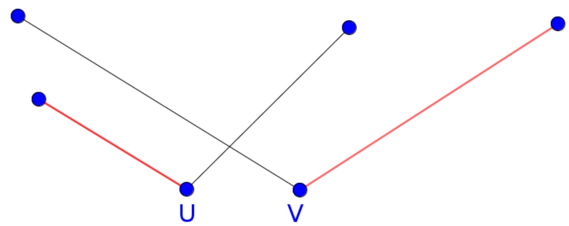

Figure 12: A drawing with two split points

Lemma 4.2. *Consider a vertex v which is a central neighbor of a fork point f. The edge $e_{v,f}$ is the only edge leaving v in the direction of f.*

Proof. Assume for the sake of contradiction that there is a vertex v which is a central neighbor of a fork point f and there is an edge $e_{v,u}$ which leaves v in the same direction as $e_{v,f}$. Without loss of generality, we may assume that $e_{v,u}$ leaves v above $e_{v,f}$. However, since v is a central neighbor of f, there is an edge $e_{f,g}$ which leaves f below the edge $e_{v,f}$, and this edge cannot cross $e_{v,u}$. This contradicts the fact that the drawing is an x-monotone thrackle. □

Lemma 4.3. *Every fork point is either a split point or adjacent to a split point. If there is a fork point which is not a split point, the split point must be a central neighbor of the fork point.*

Proof. We show that a fork point which is not a split point is adjacent to a split point. Consider a fork point v which has three vertices of degree at least 2 incident to it from one direction. Let u be a central neighbor of v. Vertex u has at least one other neighbor, but by Lemma 4.2 it must be on the opposite side of u from v. Thus, u is a split point, as shown in Figure 13. □

Corollary 4.1. *A split point may have at most two edges of degree at least two incident to it from each direction.*

Proof. If a split point has more than two edges of degree at least two incident to it in one direction, it would be adjacent to another split point, contradicting Lemma 4.1. □

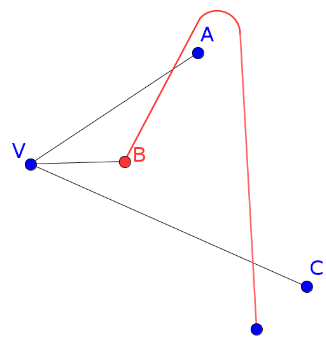

Figure 13: A fork point, v, which is not a split point, but is adjacent to the split point b

Lemma 4.4. *There are at most two fork points in any x-monotone thrackleable graph.*

Proof. We assume for the sake of contradiction that there is an x-monotone thrackle with three fork points. Lemma 4.3 tells us that either all of the fork points are neighbors of a single split point or one of the fork points is a split point and the others are its neighbors. In the first case, let the fork points be called f_1, f_2, and f_3, and call the split point s. Without loss of generality, we assume that f_2 and f_1 are on the same side of s. By Lemma 4.3, we know that s is a central neighbor of f_1. However, by Lemma 4.2, this means that s can have no neighbors other than f_1 on the same side as f_1. This contradicts the fact that f_2 and f_1 are on the same side of s.

For the second case, call the split point s and the two fork points f_1 and f_2. The proof for the first case shows that f_1 and f_2 must lie on opposite sides of the split point. Since s is a central neighbor of both f_1 and f_2, Lemma 4.2 tells us that there may be no vertex adjacent to s other than f_1 and f_2. But this contradicts the assumption that s is a fork point, so we are done. $\square$

Lemma 4.5. *No vertex may have more than three vertices of degree at least two incident to it from the same direction.*

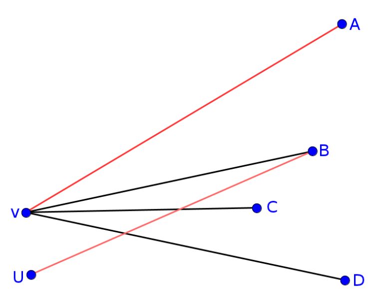

Figure 14: Four vertices of degree two adjacent to one vertex, v

Proof. Assume for the sake of contradiction that such a configuration is possible. Let the vertex under consideration be v. Call the four edges vertices adjacent to it a, b, c, and d, and let v be a left vertex, as in Figure 14. Without loss of generality, assume that, in the vicinity of v, the edges connecting v to b and c lie in between the edges connecting v to a and d. Also, assume that b is at least as far to the right as c. Let the neighbor of b which is not v be u. To intersect with $e_{v,c}$, edge $e_{b,u}$ must begin below $e_{v,b}$ and head towards the left. However, this means that there is no way for $e_{b,u}$ to intersect $e_{v,a}$, which contradicts the fact that the graph is a thrackle. ☐

Now, we prove several theorems about x-monotone thrackles.

First, we characterize x-monotone thrackles which do not contain a split point with the following theorem.

Theorem 4.1. *Any connected component of an x-monotone thrackle without a split point must be an open caterpillar.*

Proof. If there is no split point, then Lemma 4.3 tells us that there is also no fork point. Additionally, any graph with no split point is bipartite under the partition into left and right vertices, so no such graph may contain an odd cycle. Additionally, a result by Pach and Sterling [5] tells us that an x-monotone thrackle may contain no even cycle. With these facts, we may use the same argument used in Section 3 to prove that outerplanar thrackles without an odd cycle are open caterpillars. ☐

Now, we define a new type of graph which will help us characterize x-monotone thrackles.

Definition 4.6. *A* cat *is a graph which contains a single odd cycle, two paths branching off from the cycle, and any number of degree one vertices adjacent to vertices of the cycle or paths. We call the two paths* ears*. We define a k-*cat *to be a cat in which the minimum distance along the cycle between the beginnings of the two paths is k.*

An example is shown in Figure 15.

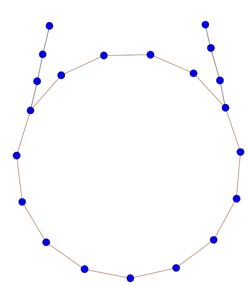

Figure 15: A 5-cat without any extra degree one vertices

Theorem 4.2. *Every connected component of an x-monotone thrackle which is a 0-cat, a 1-cat, a 2-cat, or a subgraph thereof. Additionally, if it is a 2-cat, the vertex between the two ears in the cycle may have no neighbors outside the cycle.*

Proof. If there is no split point in the component, then the graph is an open caterpillar by Theorem 4.1.

If there is a split point and an odd cycle, we present the following argument: First, we note that every vertex in the cycle already has two neighbors of degree at least two. Thus, any vertex which has a third neighbor of degree at least two is immediately a fork point, and, as such, must be a split point or adjacent to a split point, by Lemma 4.3. Also, we note that any odd cycle must contain a split point, since a graph without a split point is bipartite between left and right vertices. Now, we divide into cases by the fork points of the graph. Lemmas 4.1 and 4.4 tell us that the following are the only possible cases.

Case 1: There is only one fork point.

Call the fork point f. In this case, f must be in the cycle, as otherwise there would have to be another fork point in the cycle to connect the cycle to f. Since f is in the cycle, it already has two neighbors of degree two in the cycle. Since Lemma 4.5 and Corollary 4.1 imply that no vertex may have more than four neighbors of degree two, f can have at most two such neighbors outside the cycle, and it follows that it can have at most two paths branching off of from it. Thus, the graph is a 0-cat.

Case 2: There are two fork points, and one of them is also the split point.

Call the split point s and the fork point f. First, note that s may have no edges other than $e_{s,f}$ leaving it in the direction of $e_{s,f}$, by Lemma 4.2. As s has one edge leaving it in each direction within the cycle, $e_{s,f}$ must be an edge of the cycle, and so f must be in the cycle. Now, by Corollary 4.1 and Lemma 4.5, we know that each of s and f can have at most one neighbor of degree at least two outside the cycle. Thus, the graph must be a 1-cat.

Case 3: There are two fork points, and neither is a split point.

Call the split point s and the fork points f_1 and f_2. Since the split point is not a fork point, it can have at most two neighbors of degree at least two. As it has two such neighbors in the cycle, and two neighbors which are fork points, the fork points must be its neighbors in the cycle. Now, Lemma 4.5 implies that fork points which are not split points can have at most three neighbors of degree at least two. Since the fork points each have two such neighbors in the cycle, they may have only one such neighbor outside the cycle, so the graph must be a 2-cat. Additionally, since the split point is a central neighbor of both f_1 and f_2, it may have no incident edges other than (s, f_1) and (s, f_2).

This concludes the proof for graphs with a split point and an odd cycle. We may use the same arguments for graphs with a split point but no odd cycle to show that such graphs are subgraphs of those described above. The odd cycle is simply replaced everywhere by a path which is open on both ends. □

With all of these theorems about the connected components of an x-monotone thrackle, we may characterize x-monotone thrackles in general by the following theorem.

Theorem 4.3. *An x-monotone thrackle may have at most one component which is not an open caterpillar, and this component must be of one of the forms described in Theorem 4.2.*

Proof. By Theorem 4.1, any component of an x-monotone thrackle which is not an open caterpillar contains a split point. Since there is at most one split point in any x-monotone thrackle, there can be at most one component which is not an open caterpillar. Theorem 4.2 describes all possible components of an x-monotone thrackle which are not open caterpillars, so our proof is complete. □

So far, we have shown that, if a graph is an x-monotone thrackle, it must be of the form described in Theorem 4.3. We now proceed to prove the converse by demonstrating constructions for all such graphs. First, we prove two lemmas.

Lemma 4.6. *Given an x-monotone thrackle with a vertex v, we may add a degree one neighbor of v to the graph if there is some vertex u in the graph of which v is a neighbor, but not a central neighbor.*

Proof. Consider a vertex v in an x-monotone thrackle which has a neighbor u of which v is not a central neighbor. Without loss of generality, assume that $e_{u,v}$ leaves u above all other edges leaving u in its direction. Let the coordinates of u in the plane be (x, y). Then, to add a degree one vertex to v, we place a construction point, P, at coordinates $(x, y-h)$, and the vertex, w, at coordinates $(x-h^2, y+h)$, for some arbitrarily small positive h. We then draw $e_{v,w}$ as the union of the segment vP, which is drawn from v to P just below $e_{u,v}$, and segment Pw, which is drawn as a straight line. This is shown in Figure 16. We see that $e_{v,w}$ intersects $e_{u,v}$ at v and all edges which are not incident to u along the segment vP. Segment vP also intersects all of the edges leaving u to the right, and segment Pw intersects all of the edges leaving u to the left. Thus, $e_{v,w}$ intersects every edge in the graph exactly once, so the graph is still an x-monotone thrackle. □

Lemma 4.7. *Given an x-monotone thrackle with a degree one vertex v adjacent only to a degree two vertex u, we may add a single path of any length to v.*

Proof. We proceed by induction on the length of the path.

Base case: We note that, since u is of degree two, v cannot be a central neighbor of u, so by Lemma 4.6 we may add a degree one vertex as a neighbor of v. This is a path of length one.

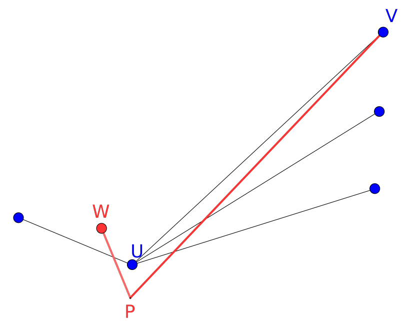

Figure 16: A degree one vertex, w, added to the vertex v

Inductive step: Given a path of length k, the end always has a degree one vertex v adjacent only to a degree two vertex. Thus, by Lemma 4.6, we can add a degree one vertex as a neighbor of v, obtaining a path of length $k+1$. $\square$

With these lemmas in hand, we now demonstrate constructions of the three allowed types of cats.

Theorem 4.4. *Any* 0*-cat, any* 1*-cat, and any* 2*-cat that has no leaf adjacent to the split point is* x*-monotone thrackleable.*

Proof. First, we construct a 0-cat. To do this, begin with the odd cycle as a star, which we will consider to be oriented as shown in Figure 17. Then, because s is not a central neighbor of either of its neighbors, we may by Lemma 4.6 add a degree one vertex a which is adjacent to s on the left and a degree one vertex b which is adjacent to s on the right. Now, neither a nor b is a central neighbor of s, so we may add two more degree one vertices: one adjacent to a and one adjacent to b. At this point, we have two vertices of degree one adjacent only to vertices of degree two, so Lemma 4.7 tells us that we may add a path of any length to each one. This will give us the cycle and ears of a 0-cat. Now, because the only vertex in the graph which has more than two neighbors is the split point, and no vertex is adjacent to only the split point, every vertex has a neighbor of which it is not a central neighbor. Therefore, we may add any number of degree one vertices to any vertex in the graph and obtain every possible 0-cat.

To create a 1-cat or a 2-cat, the proof is the same except for the construction of the initial two vertices of the ears and for the addition of degree one vertices in the 2-cat case. To create the beginning of the path originating from a neighbor v of the split point s, we first draw an edge which travels below $e_{s,f}$, under s, and up through the other edge incident to s. The next edge follows this same path back just above the first edge, but cuts up through both neighbors of v at the end. These constructions are shown in Figures 18 and 20. The only other important point is that, in a 2-cat, after constructing the ears and the cycle, the split point has no neighbors of which it is not a central neighbor. Thus, it cannot have any degree one vertices added to it. □

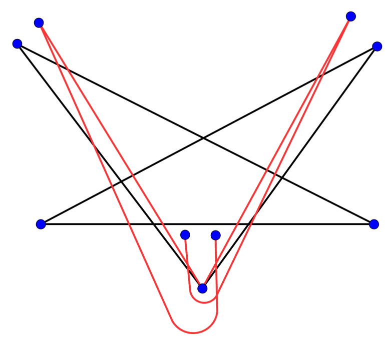

Figure 17: A construction of a 0-cat

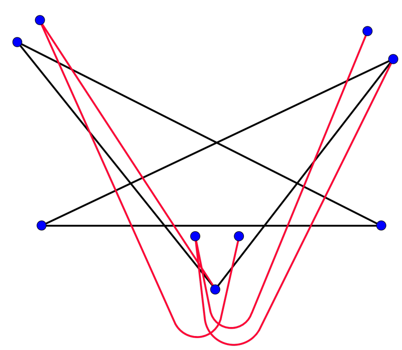

Figure 18: A construction of a 1-cat

Now, all that is left is to show that we may combine the constructions of the connected components in the ways that we claim are possible.

Theorem 4.5. *Any combination of some number of open caterpillars and at most one component of the types described by Theorem 4.2 is* x*-monotone thrackleable.*

Proof. We will demonstrate how to construct every such combination. We first remember our construction of a set of open caterpillars for outerplanar thrackles. We see that, by moving the regions $a_1, a_2, \ldots, a_n$ into a single very small region, and the regions $b_1, b_2, \ldots, b_n$ into a different very small region, we flatten out the construction so that, to edges outside of the closed caterpillars, the whole set of caterpillars is equivalent to a single edge. This is shown in Figure 20.

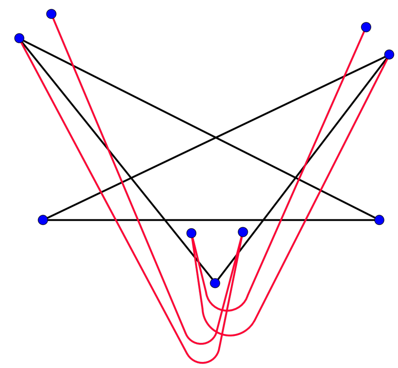

Figure 19: A construction of a 2-cat

To complete the proof, we must show that we may add one disjoint edge to any x-monotone thrackle which is not an open caterpillar. We divide into two cases.

Case 1: The graph is a cat with an ear of length at least two. In this case, we may find a vertex v which is of degree one, which is not a central neighbor of its neighbor u, and which is not adjacent to the graph's split point. Assume without loss of generality that $e_{v,u}$ leaves u above all other edges incident to u. Then, to draw a segment disjoint from the graph which intersects every edge in the graph, we begin just above v, follow the path of $e_{v,u}$, crossing it once along the way, and end just below vertex u.

Case 2: The graph has no ear of length at least two. In this case, the graph must be a closed caterpillar. Call the split point s, and let the top edge leaving s on the right connect s to the vertex a. To draw a segment disjoint from the graph which intersects every edge in the graph, we start just above a, follow a straight line path to a point very close to and directly below s, and then follow a nearly vertical line upward to hit all of the edges which are incident to s on the other side, as shown in Figure 21. This completes the proof. $\square$

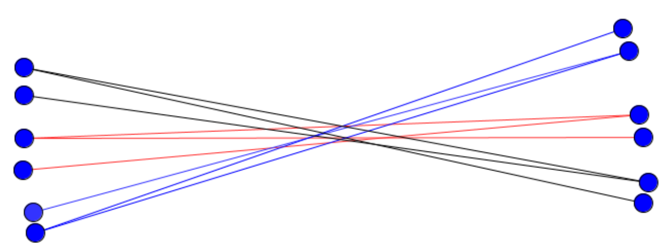

Figure 20: A set of open caterpillars, flattened out to act like a single edge

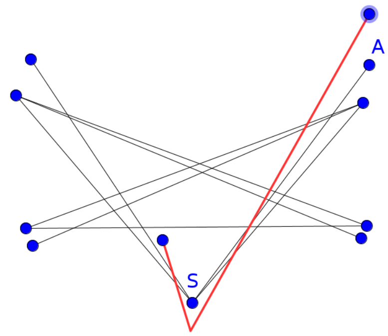

Figure 21: A closed caterpillar with a disjoint edge

5 Conclusion

We examined two special cases of thrackles. We first characterized outerplanar thrackles based on one simple impossible configuration. We then characterized x-monotone thrackles by a more in-depth analysis involving several special types of vertices and some casework. In the future, this work may help give ideas for how to approach the general thrackle problem, which, if solved, will likely help in turn to give insight into a broad range of related problems in graph drawings.

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