PRIME STEP *Plays Games*

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In the famous mathematical strategy game Nim, two players face a certain number of heaps, each containing some number of stones. Each turn, a player takes one or more stones from one heap. The player with no move loses. (There is also a *misère* version in which the player who is forced to take the last stone loses.)

Nim has been completely solved: Given any collection of stones, we can say which player will win, assuming the player follows a winning strategy, and we know the winning strategy. All Nim-like games—technically called *impartial combinatorial games*—have the property that there is a winning strategy for one of the players, although we may not know the winning strategy.

This article describes some two-person impartial combinatorial games that were invented and analyzed by students in grades seven through nine who were mentored by Tanya Khovanova. Each game has its own flavor.

Rows-and-Columns Game

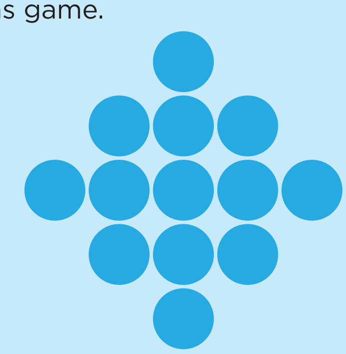
In this game, tokens are placed on intersections of grid lines on a square grid. They are not treated as heaps like in Nim. The geometry of the configuration plays a role. A player is allowed to take any nonempty row or column of tokens. The player unable to take a token loses.

As often happens, the simplest case has symmetries. If the shape does not contain tokens on the axes and is symmetric with respect to both axes, then player 2 has a winning strategy: If player 1 makes a horizontal move, player 2 mirrors the move with respect to the x-axis. If player 1 makes a vertical move, player 2 mirrors the move with respect to y-axis. After two moves, the shape is symmetric with respect to both axes again, and all the tokens cannot be removed in one move.

Now, consider a diamond configuration in which tokens are put on points with integer coordinates a and b, such that $|a|+|b|\leq c$. See figure 1 for an example with $c=2$. Player 1 wins when the diamond consists of one token. For larger diamonds, player 2 is guaranteed a win using the following strategy. First, the end game: If the tokens form a line, take the line. If the tokens form two lines in one axial direction, take a line in the other direction. In the middle of the game: If the other player takes one axis, take the other axis. Otherwise, if the other player takes a line parallel to axis a, take the mirror image of the line with respect to a.

In the study of impartial combinatorial games, a *P-position* is a game configuration from which the *previous* player—that is, the player who just played or player 2 at the start of the game—wins given perfect play. All the terminal positions are

Figure 1. A diamond in the Rows-and-Columns game.



P-positions. An *N-position* is a configuration from which the *next* player wins given perfect play. For example, unless it has one token, a diamond configuration is a P-position because player 2 can always win. A player wants to end her turn with the game in a P-position and wants to see an N-position before her move.

Many games are solved by first providing the sets of P- and N-positions. To prove that they are correct, it is enough to show that *there exists* a move from an N-position to a P-position and that *any* move from a P-position is to an N-position.

Consider the Rows-and-Columns game, in which the tokens form a cross on both axes, including the center (see figure 2). The game depends only on the number of tokens on each axis, which we denote m and n. A cross is a P-position provided $m + n$ is even and $m, n > 1$. To prove this, observe that no player wants to remove an axis when the other axis has more than one token. So, each player removes one token per move so that the remaining tokens do not form a line. This continues until the players get to the cross with $m = n = 2$. Then, player 2 wins.

Remove-a-Square Game

This game is played on a square grid. Start with a shape made out of 1×1 squares. Each move, a player removes a $k\times k$ square for some k. The player who has no squares to take loses.

We will focus on the $2\times n$ rectangle game. In this case, each turn a player can remove a 1×1 or a 2×2 square. For instance, suppose the game starts with a 2×3 rectangle. If player 1 removes a 1×1 square, player 2 can remove a 1×1 square from the middle column. After that, there are no 2×2 squares, and player 2 wins in six moves. On the other hand, if player 1 removes a 2×2 square, the game ends in three moves, and player 1 wins.

If we start with a $2\times 2k$ rectangle, player 1 has the following winning strategy: Remove the middle 2×2 square. If $k > 1$, the game separates into two identical configurations. Then, for each of player 2's moves on one board, make the identical move on the other board.

To solve the general $2\times n$ game, we will assign to each game a nonnegative integer called a *Grundy number*. Grundy numbers have the property that they are 0 exactly for the P-positions. We assign Grundy numbers recursively. Consider a position A. All moves from A comprise a set of positions. Let S be the set of Grundy numbers for these positions. The Grundy value of A is the least nonnegative integer not in S; we denote this *minimum excluded value* $\text{mex}(S)$.

For example, consider the five-block shown at the top of figure 3. There are six possible moves: There's one way to remove a 2×2 block and five ways to remove a 1×1 block. Each resulting shape can be decomposed, and we obtain the graph shown in the figure. We then work our way from the bottom up. The terminal position at the bottom has Grundy number 0. The single block on the next level has Grundy number $\text{mex}(\{0\}) = 1$. The two 2-block configurations on the next level have Grundy number $\text{mex}(\{1\}) = 0$. Continue in this way up to the top, concluding that that configuration has Grundy number $\text{mex}(\{0,1,2\}) = 3$.

We can assign Grundy numbers to any such impartial combinatorial game. A cool fact, which we will not elaborate on, is that two games with the same Grundy numbers are equivalent. This is the Sprague–Grundy theorem.

There's a surprising arithmetic trick that enables us to compute Grundy numbers. To understand it, we must introduce the notion of

Figure 2. This cross is a P-position for the Rows-and-Columns game.

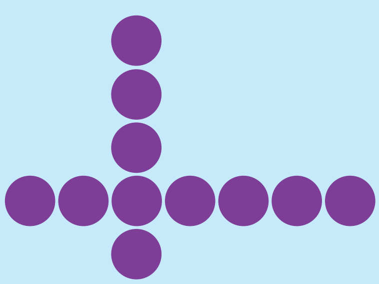
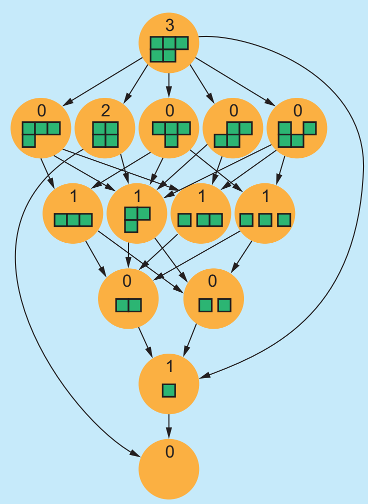


Figure 3. Using a recursive algorithm, we compute that the Grundy number of the five-box game is 3.



XOR or *bitwise addition*, which we denote $\otimes$. To compute, say, $13 \otimes 17$, we convert 13 and 17 into binary; they are 1101 and 10001, respectively. Then we add them using the XOR operation. In other words, we add them so that $0+0=0$, $1+1=0$, and $0+1=1$. In this case, we obtain 11100, or 28:

$$\begin{array}{rccccc} & & 1 & 1 & 0 & 1 \\ \otimes & 1 & 0 & 0 & 0 & 1 \\ \hline & 1 & 1 & 1 & 0 & 0 \end{array}$$

If we can decompose a game into a sum of games, the Grundy value of the game is the bitwise sum of the Grundy values of the smaller games. For instance, suppose our board is a 1×1 and a 2×2 square, which have Grundy values 1 and 2, respectively. Then the Grundy value of the game is $1 \otimes 2 = 3$. (In binary, $1 \otimes 10 = 11$.)

We can now return to our $2 \times n$ game. In the first move, player 1 can remove a 2×2 square or a 1×1 square. The first move leaves the sum of games on $2 \times k$ and $2 \times (n-k-2)$ rectangles (one of which could be the empty game), and the second leaves the sum of games on $2 \times k$, 1×1, and $2 \times (n-k-1)$ rectangles. This allows us to calculate the Grundy values recursively.

Let's denote the Grundy number of a $2 \times n$ rectangle as $G(n)$. Then $G(0) = G(1) = 0$ and

$$G(n) = \text{mex}(\{G(i) \otimes G(n-i-2), G(j) \otimes 1 \otimes G(n-j-1)\})$$

where $i = 1, \ldots n-2$ and $j = 1, \ldots, n-1$. For instance, figure 4 shows the 2×3 case, in which

$$\begin{aligned} G(3) &= \text{mex}(\{G(1) \otimes G(0), G(1) \otimes 1 \otimes G(1), G(2) \otimes 1 \otimes G(0)\}) \\ &= \text{mex}(\{0 \otimes 0, 0 \otimes 1 \otimes 0, 2 \otimes 1 \otimes 0\}) \\ &= \text{mex}(\{0,1,3\}) \\ &= 2. \end{aligned}$$

Using this recurrence, we computed the first 200 values of $G(n)$ (starting with $n=0$) and displayed the first 96 in table 1. We presented them in 12 columns to emphasize the eventual periodicity. We will prove that the periodicity begins at $n = 71$. This sequence of Grundy numbers is now sequence A286332 in the Online Encyclopedia of Integer sequences (OEIS). See *oeis.org*.

It follows that for $n > 0$, the P-positions are periodic: They correspond to numbers $n = 12a + 1$. In other words, under perfect play, the second player can win $2 \times n$ games for these n. In all other cases, player 1 can win.

We will prove the periodicity result by induction. We verified that $G(k) = G(k-12)$ for $k = 83, \ldots, 167$. These are the base cases. Now, assume $n \geq 168$ and $G(k) = G(k-12)$ for all $k = 83, \ldots, n-1$. We claim that the set of Grundy values obtained from any move on a $2 \times n$ or a $2 \times (n-12)$ game are the same, and thus $G(n) = G(n-12)$. If we remove a 2×2 square from a $2 \times n$ rectangle, the resulting game is the sum of games on $2 \times k$ and $2 \times (n-k-2)$ rectangles for some $0 \leq k \leq n-2$. The Grundy value for this game is $G(k) \otimes G(n-k-2)$. Either k or $n-k-2$ is at least 83; we may assume that k is. By our inductive assumption,

$$G(k) \otimes G(n-k-2) = G(k-12) \otimes G(n-k-2),$$

and we can obtain this value by removing a 2×2 square in a game on a $2 \times (n-12)$ rectangle. Conversely, any value obtained by removing a 2×2 square in a game on a $2 \times (n-12)$ rectangle corresponds to one obtained by removing a 2×2 square from a $2 \times n$ rectangle. A similar equivalence holds if we remove a 1×1 square from either rectangle. Thus, $G(n) = G(n-12)$.

Remove-an-Edge Game

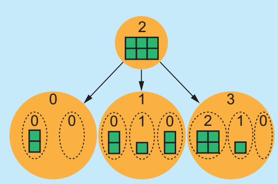
This game is played on a simple graph (one without loops or parallel edges). Each turn, a player is allowed to remove two neighboring vertices and all edges coming out of them. The player who does not have a turn loses.

Consider a star graph with $n > 1$ vertices (see figure 5). Player 1 can win in one move by removing the central vertex, one other vertex, and all edges, leaving $n-2$ isolated vertices.

Table 1. The first 96 Grundy numbers $G(n)$.

0	0	2	2	1	4	3	3	1	4	2	6
5	0	2	7	1	4	3	3	1	4	7	7
5	0	2	8	4	4	6	3	1	8	7	7
5	0	2	2	1	4	6	3	1	8	2	7
5	0	2	8	1	4	6	3	1	4	2	7
5	0	2	8	1	4	6	3	1	8	7	7
5	0	2	8	1	4	6	3	1	8	2	7
5	0	2	8	1	4	6	3	1	8	2	7

Figure 4. We can use XOR to compute $G(3)$.



If we begin with a complete graph (every pair of vertices are connected by an edge) with $n > 1$ vertices, any move turns the game into a complete graph with $n - 2$ vertices. This is a no-strategy game. Player 1 wins if n equals $4k + 2$ or $4k + 3$ for some k. Otherwise, player 2 wins.

Let P_n be a path graph with n vertices. This game is equivalent to a known game called *domino covering* on a $1 \times n$ rectangle. In the domino-covering game, the players take turns placing dominoes (1×2 rectangles) on a board that is a $1 \times n$ rectangle. The loser is the first player unable to move. The P-positions in this game are OEIS sequence A215721. It begins 0, 1, 5, 9, 15, 21, 25, 29, 35, 39, 43, 55, 59, 63, and then afterward $\text{A215721}(n) = \text{A215721}(n-5) + 34$. In particular, other than 0, all P-positions are odd. If n is even, player 1 can remove the center edge, dividing the game into two equivalent games, and player 1 wins.

The Grundy numbers for this game can be computed recursively: After one move, a path graph becomes the union of two smaller path graphs. If we let $G(n)$ denote the Grundy value for the $1 \times n$ rectangle, then $G(0) = G(1) = 0$ and

$$G(n) = \text{mex}(\{G(i) \otimes 1 \otimes G(n-i-2)\}),$$

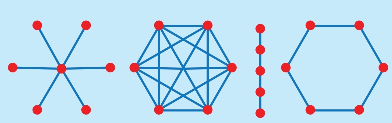
where $i = 1, \ldots, n-2$.

Next, consider a cycle graph with n vertices, C_n. After the first move, the game on C_n is equivalent to the game on the path graph P_{n-2}. So, the N-positions are $\text{A215721}(n) + 2$. This is sequence A274161 in the OEIS.

The OEIS describes A274161 in terms of a different game—the *edge-delete game*—in which two players alternate turns, permanently deleting one edge from a graph. Unlike our game, the vertices are not removed. The game ends when a vertex is isolated. The player whose deletion creates an isolated vertex loses.

Sequence A274161 gives the P-positions of the edge-delete game played on P_n. Let us consider a domino-covering game on the $1 \times (n+2)$ rectangle. We can associate a vertex to each square cell and connect the vertices if the cells share an edge. Then, deleting an edge corresponds to covering the vertices connected by this edge by a domino. The rule that does not allow isolated vertices means that the dominoes cannot overlap and cannot cover the ending cells. So, the edge-delete game on P_n is the same as the domino-covering game on a $1 \times n$ rectangle, where dominoes can't cover the end squares. The latter game is the same as the domino-covering game on the $1 \times (n-2)$ rectangle without end restrictions. Consequently, the sequence of P-positions of the game on C_n is the complement of sequence A274161.

Figure 5. Left to right: a star graph, a complete graph, a path graph, and a cycle graph.



No-Factor Game

Write out the integers 1 through n. Each turn, a player may remove any set of numbers that have no proper factors existing at the beginning of the turn. The person who does not have a move loses.

This game can be solved using a "strategy stealing" argument. Surprisingly, it is player 2 who is stealing. We will show that for $n > 1$, player 2 wins the no-factor game.

Player 1 must take 1. If $n = 1$, player 1 wins. Suppose $n > 1$ and, for the sake of contradiction, player 1 has a winning strategy. Suppose player 2 was to take a prime number $p > n/2$; such a prime exists due to a theorem called Bertrand's postulate. By assumption, this is an N-position. Player 1's winning strategy would prescribe a certain move that would put the game in a P-position. But p does not have any common factors with the rest of the numbers, so on the second move, player 2 can take p and whatever numbers player 1 would have taken on the third move. This leaves the game in a P-position after player 2's turn, thereby stealing the winning strategy. This is a contradiction. Player 2 has a winning strategy.

Further Reading

A longer version of this article containing more games can be found at *arxiv.org/abs/1707.07201*. See *Winning Ways for Your Mathematical Plays* by Elwyn Berlekamp, John Conway, and Richard Guy (AK Peters, 2001) or *Lessons in Play* by M. H. Albert, R. J. Nowakowski, and D. Wolfe (AK Peters, 2007) for more on impartial games.

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