Tensor Product Decompositions for Modules Over Subregular W-algebras

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Mission San Jose High School

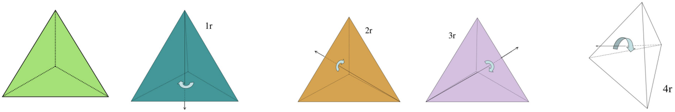
October 15, 2023

Concept of symmetry is very important in mathematics and physics (e.g. gauge theory).

- Mathematically formalized by groups

Instead of a formal definition, we give some examples of groups:

1. The symmetric group S_n, set of all permutations of n elements;
   - A bijection of a set onto itself, definition of "symmetry"
   - Symmetry group S_4 is an isometric permutation of vertices for tetrahedron:



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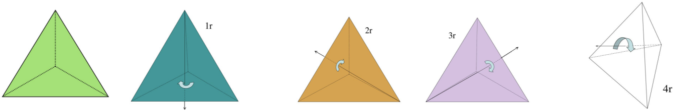

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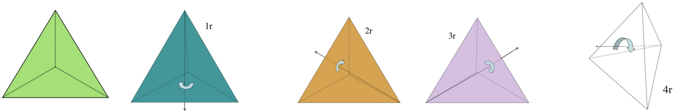

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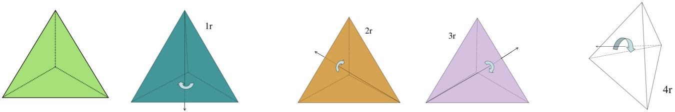

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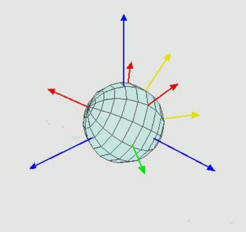
For example:

- Rotations of a sphere $\mathrm{SO}(3)$



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$$\mathrm{SL}(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| ad - bc = 1 \right\}$$

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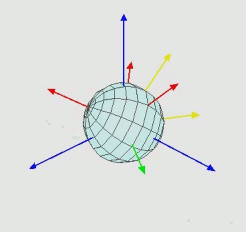

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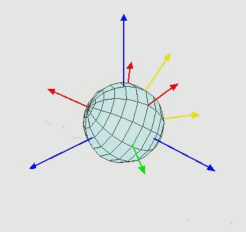

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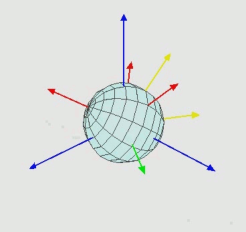

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Examples — derivations:

1. $D_1 = \frac{d}{dx}$, infinitesimal version of *translations* by Taylor's formula:

$$f(x+t) = f(x) + t \cdot df/dx + O(t^2).$$

2. $D_2 = x\frac{d}{dx}$: infinitesimal version of *dilations* $f(e^t x)$.

Observe both D_1, D_2 are derivations. However, they do not have an algebra structure:

- $D_1 \circ D_2$ is not a derivation
- But $D_1 \circ D_2 - D_2 \circ D_1 = \frac{d}{dx}$ is

We denote $D_1 \circ D_2 - D_2 \circ D_1$ by $[D_1, D_2]$.

Definition

A ***Lie algebra*** is a vector space $\mathfrak{g}$ equipped with the skew-symmetric bilinear map $[-,-]$ satisfying the *Jacobi identity*

$$[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0, \ \ a,b,c \in \mathfrak{g}.$$

Examples:

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A ***representation*** of $\mathfrak{g}$ is a vector space $\mathbb{C}^n$ with a map of Lie algebras $\mathfrak{g} \to \mathfrak{gl}_n$.

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$$\mathfrak{sl}_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a + d = 0 \right\}$$

Over complex numbers: same as the Lie algebra of $SO(3)$.
For representations, take

$$E \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad F \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad H \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

More abstractly: spanned by E, F, H with relations

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H.$$

How to study representations? Basic building block — irreducibles:

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Irreducible representations

For $\mathfrak{sl}_2$ — complete classification:

Theorem

Irreducible finite-dimensional representations V_n of $\mathfrak{sl}_2$ are classified by a natural number n and are of the form $V_n := span(v, Fv, F^2v, \ldots, F^nv)$, where the vector v satisfies $Ev = 0$ (highest-weight vector).

Natural operation on representations: tensor product (for instance, corresponds to combined system of particles)

- Decomposition of tensor products are actually completely determined by highest weight vectors

Theorem (Clebsch-Gordan)

For irreducible representations V_n, V_m, we have $V_n \otimes V_m \cong \bigoplus_{k=0}^{min(n,m)} V_{n+m-2k}$.

For instance, let $\mathbb{C}^2 = \text{span}(v_1, v_2)$ where $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then, highest-weight vectors of $\mathbb{C}^2 \otimes \mathbb{C}^2$ are

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Whittaker Modules

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- What if $Ev = v$? Called ***Whittaker vectors***, generate *Whittaker modules*. Naturally arise in physics (Toda system).
- Decomposition of Whittaker modules: likewise, completely classified by Whittaker vectors $\mathrm{Whit}(\mathcal{W})$.

Theorem (Kalmykov, 2021)

For any Whittaker module $\mathcal{W}$ and a finite-dimensional representation V of $\mathfrak{sl}_2$, we have $Whit(\mathcal{W} \otimes V) \cong Whit(\mathcal{W}) \otimes V$ canonically.

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Non-standard quantization

Application: non-standard quantization of SL_2. Two ways to compute Whittaker vectors in $\mathcal{W} \otimes U \otimes V$:

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Differ by the action on $U \otimes V$ of

$$J = \sum_k J_k^{(1)} \otimes J_k^{(2)} = \sum_{k \geq 0} \frac{(-1)^k}{2^k k!} F^k \otimes \prod_{i=0}^{k-1} (H - 2i).$$

Deforms multiplication on functions on SL_2:

$$f * g := \sum_i J_k^{(1)}(f) \cdot J_k^{(2)}(g), \;\; f, g \in \mathrm{Fun}(\mathrm{SL}_2).$$

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Generalization

Generalization: ***W-algebras*** (Whittaker Modules for $\mathfrak{gl}_n$).
Our research: tensor product decomposition for subregular W-algebras.

Theorem (Kalmykov-L., 2023)

For any subregular Whittaker module $\mathcal{W}$ and the vector representation V of $\mathfrak{gl}_n$, there is an explicit identification

$$\mathrm{Whit}(\mathcal{W} \otimes V) \cong \mathrm{Whit}(\mathcal{W}) \otimes V.$$

In particular, allows to construct canonically Whittaker vectors in $\mathcal{W} \otimes U$ for *any* finite-dimensional representation U of $\mathfrak{gl}_n$.

Likewise, gives non-standard quantization of the group GL_N.

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- My parents for always being so supportive.

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