

# Canonical Forms for Toric and Surface Codes in ZX Calculus

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MIT PRIMES Conference

# Quantum processes are noisy



Quantum error-correcting  
codes correct for noise

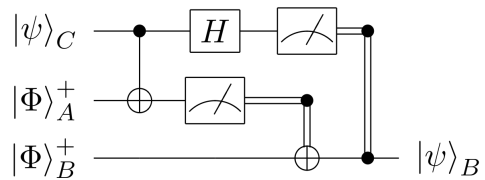


Graphical languages  
present quantum circuits  
and codes



ZX calculus simplifies and  
condenses models

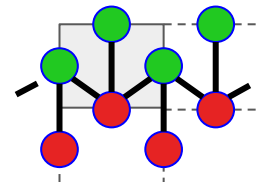
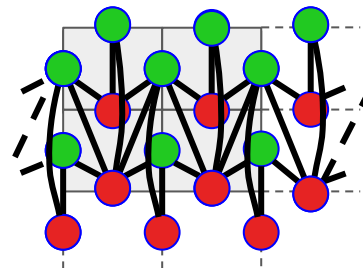
# Quantum Computing



Connects with...



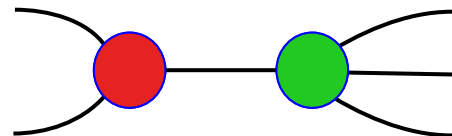
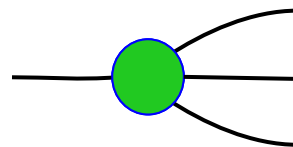
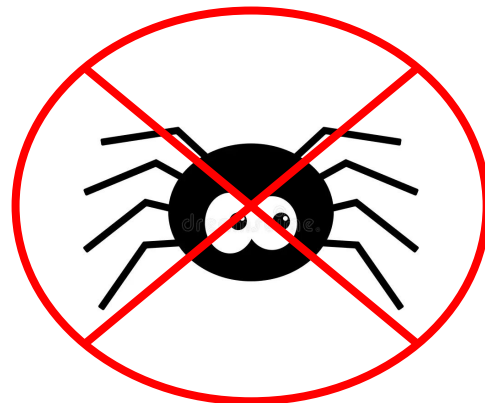
# ZX Diagrams



# Introducing ZX Calculus:

## Spiders!

- Either **red** and **green**
- Connected by edges
- Represent qubits, gates, and measurements

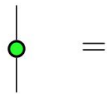
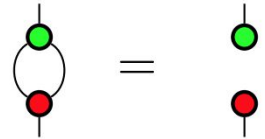
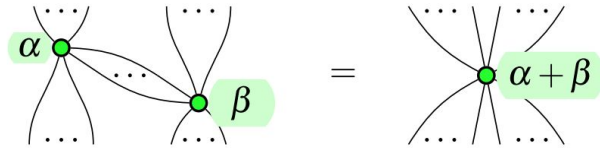


# ZX Calculus

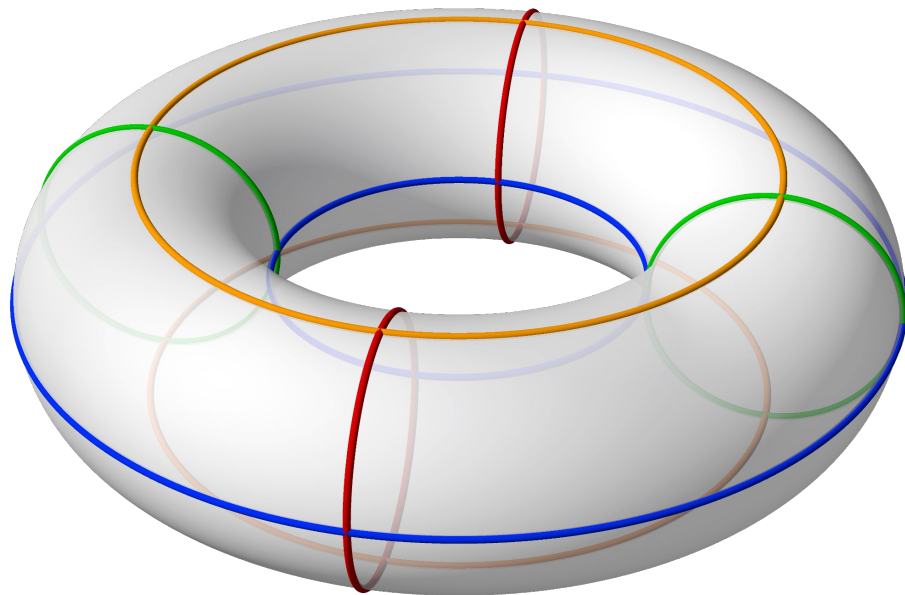
After converting a circuit to a ZX diagram, we can:

- Use rewrite rules to simplify

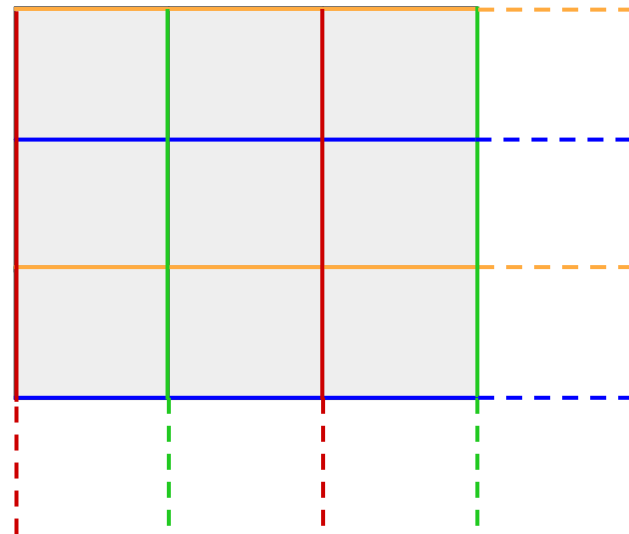
Rewrite rules apply to both colors equally



Torus

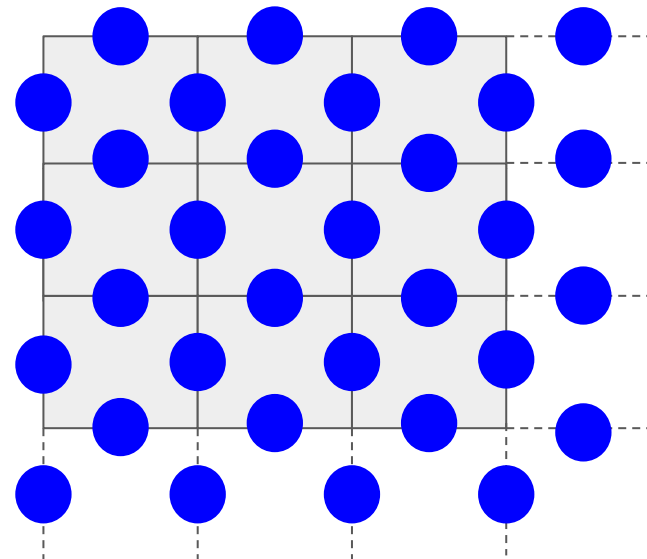


Unwrapped torus

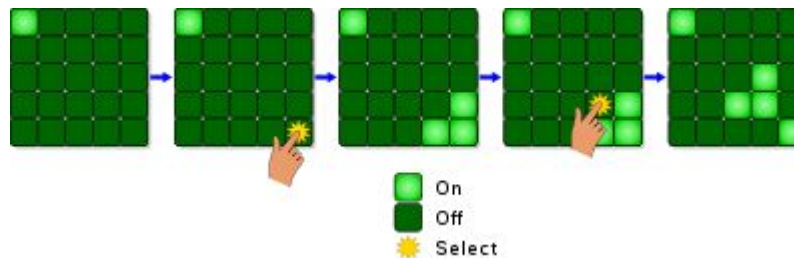


## “Toric Game” Rules:

1. Midpoint of torus’s edges are *nodes*.
2. Nodes are red or green.
3. Red nodes only connect to green nodes by edges and vice versa.
4. Each node’s light is on or off.



To illustrate the Toric code, consider the game LIGHTS OUT!  
Pressing a button toggles itself and those that are adjacent to it.



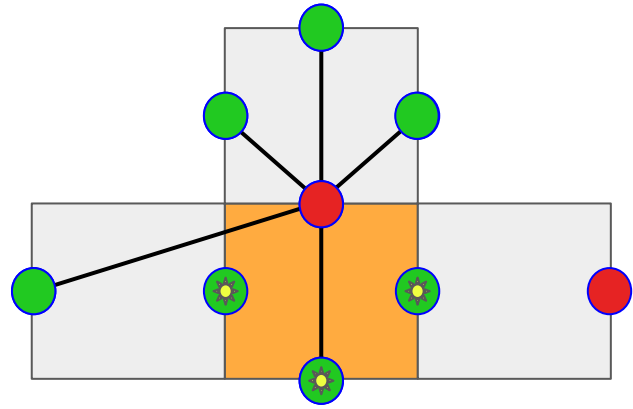
## **Toric Rules (cont.):**

5. Can toggle lights around a face or vertex.



## Toric Rules (cont.):

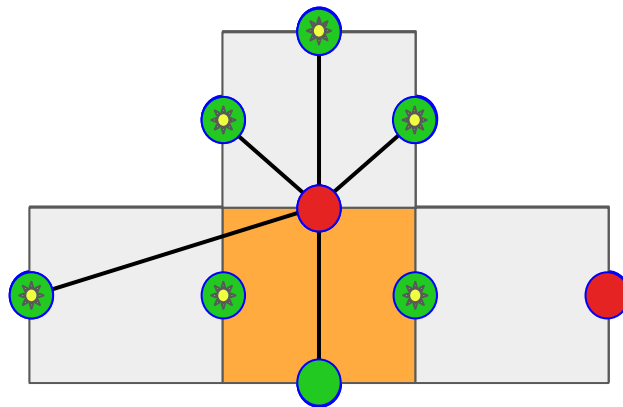
5. Can toggle lights around a face or vertex.
6. Around a face
  - Green lights turned on



Green nodes surrounding orange face turned on.

## Toric Rules (cont.):

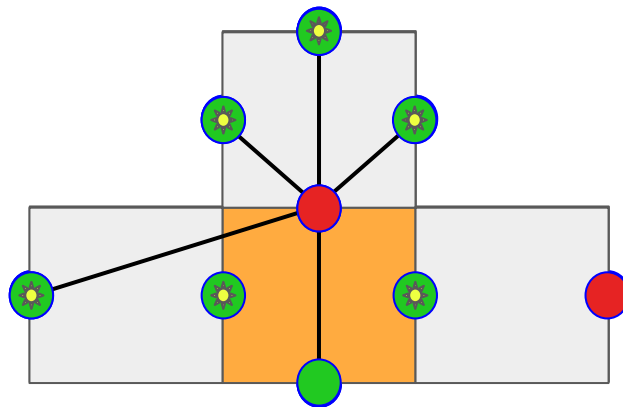
5. Can toggle lights around a face or vertex.
6. Around a face
  - Green lights turned on
  - All green neighbors of red lights toggled



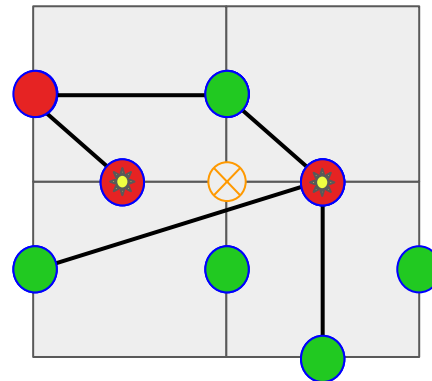
Green nodes adjacent to red node(s) surrounding the orange face are toggled.

## Toric Rules (cont.):

5. Can toggle lights around a face or vertex.
6. Around a face
  - Green lights turned on
  - All green neighbors of red lights toggled
7. Around a vertex
  - Red lights turned on

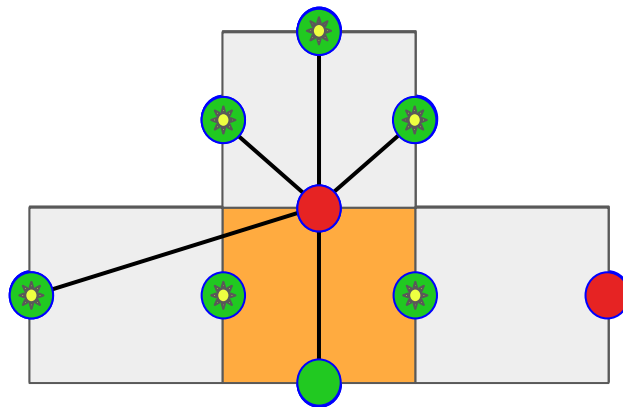


Green nodes adjacent to red node(s) surrounding the orange face are toggled.

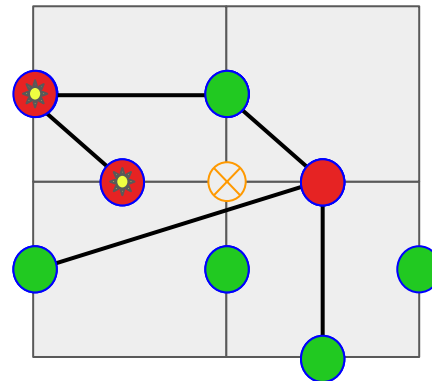


## Toric Rules (cont.):

5. Can toggle lights around a face or vertex.
6. Around a face
  - Green lights turned on
  - All green neighbors of red lights toggled
7. Around a vertex
  - Red lights turned on
  - All red neighbors of green lights toggled

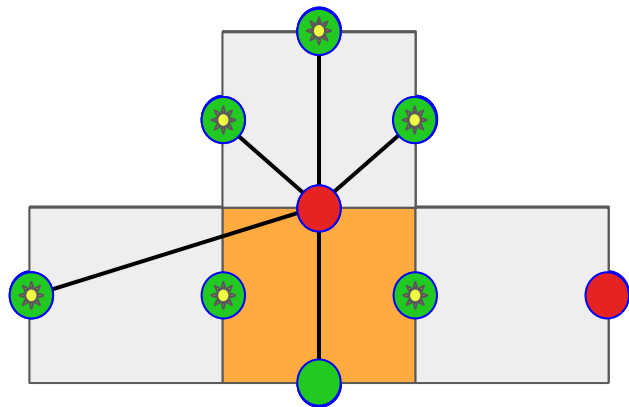


Green nodes adjacent to red node(s) surrounding the orange face are toggled.

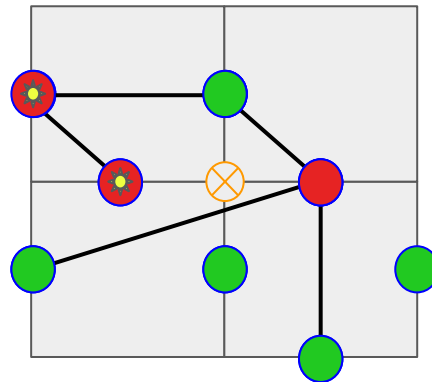


## Toric Rules (cont.):

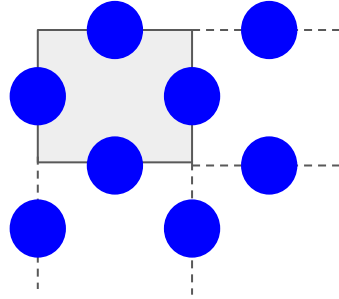
5. Can toggle lights around a face or vertex.
6. Around a face
  - Green lights turned on
  - All green neighbors of red lights toggled
7. Around a vertex
  - Red lights turned on
  - All red neighbors of green lights toggled
8. Goal: Arrange the colors and connections so any light-switching keeps all lights off.



Green nodes adjacent to red node(s) surrounding the orange face are toggled.

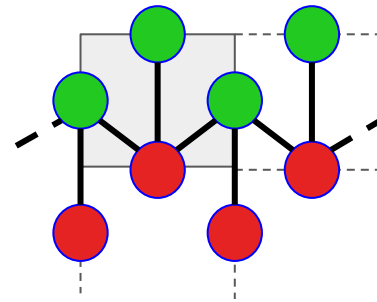
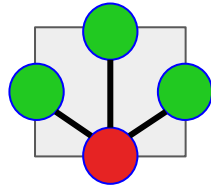


We start small, with a 2 by 2 grid of output nodes.



Result:

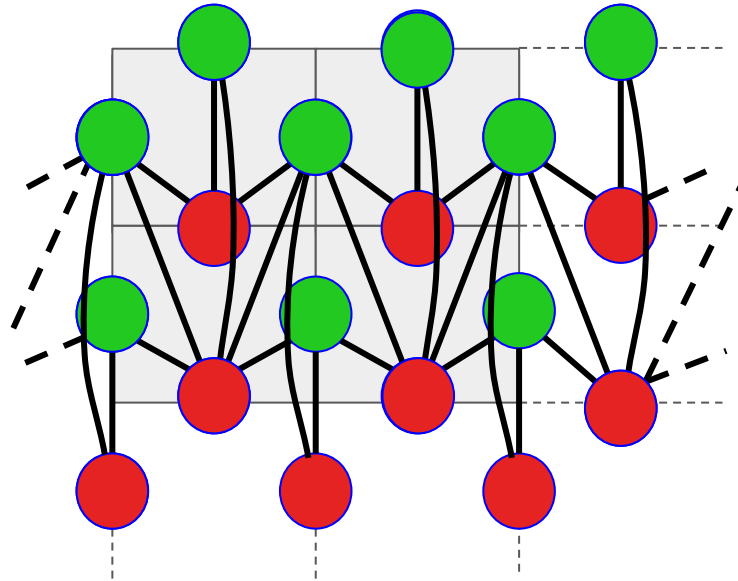
**Arrow** structure:



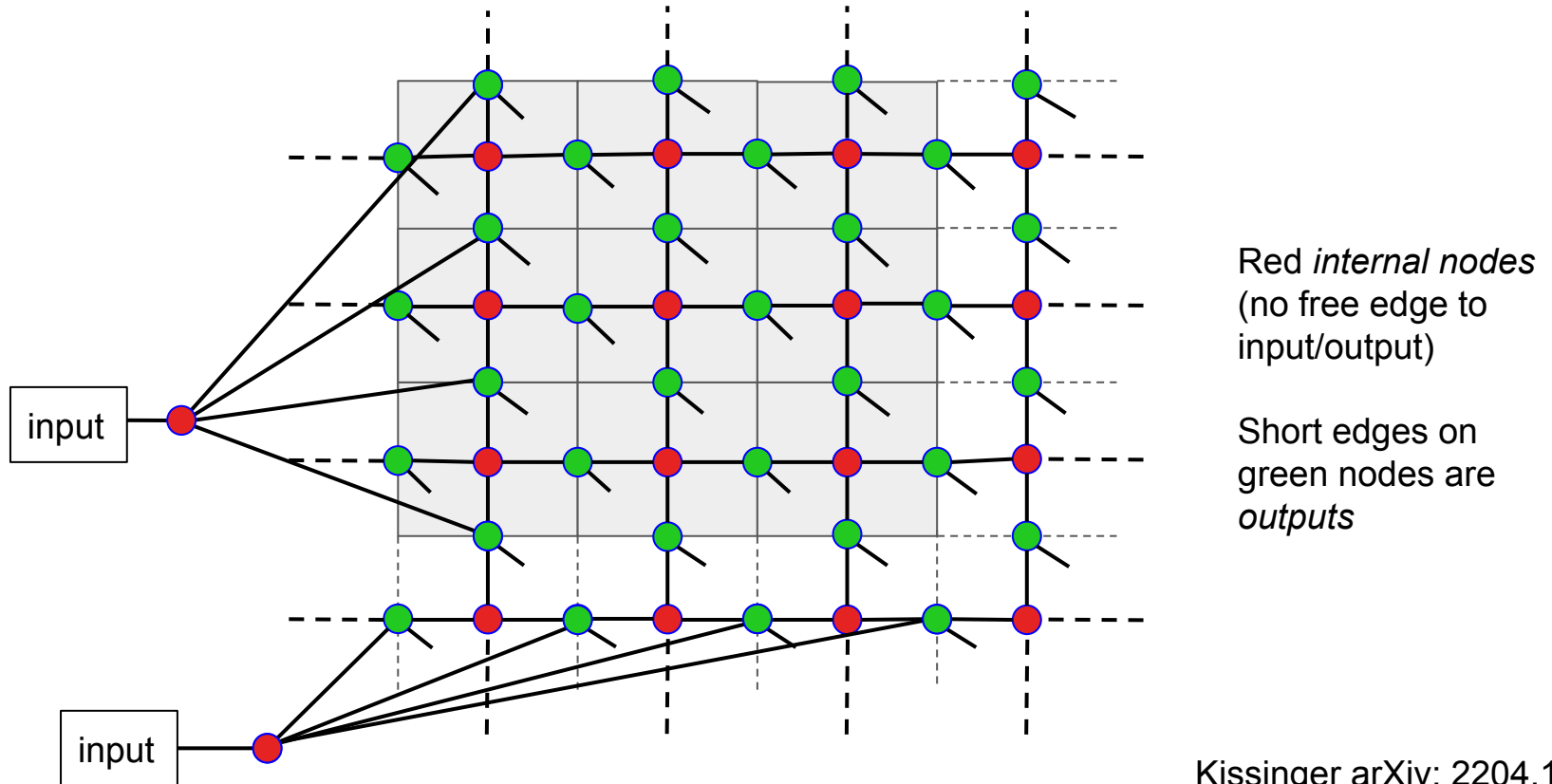
Dashed edges wrap around the torus.

We move onto a 3 by 3 grid of output nodes:

After a bit of work, we can find:

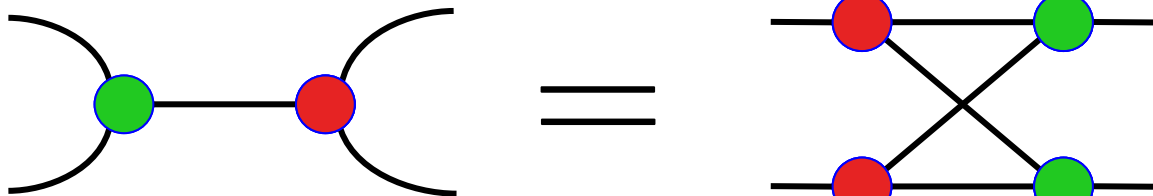


# Kissinger Normal Form for Toric Code

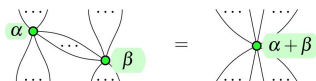
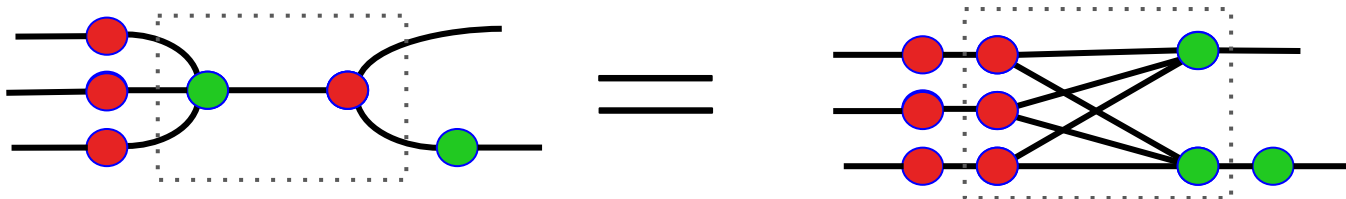




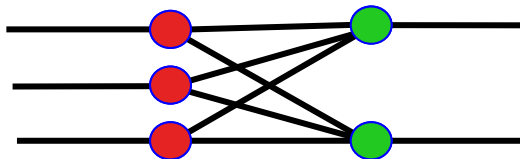
# Bialgebra rule removes internal nodes



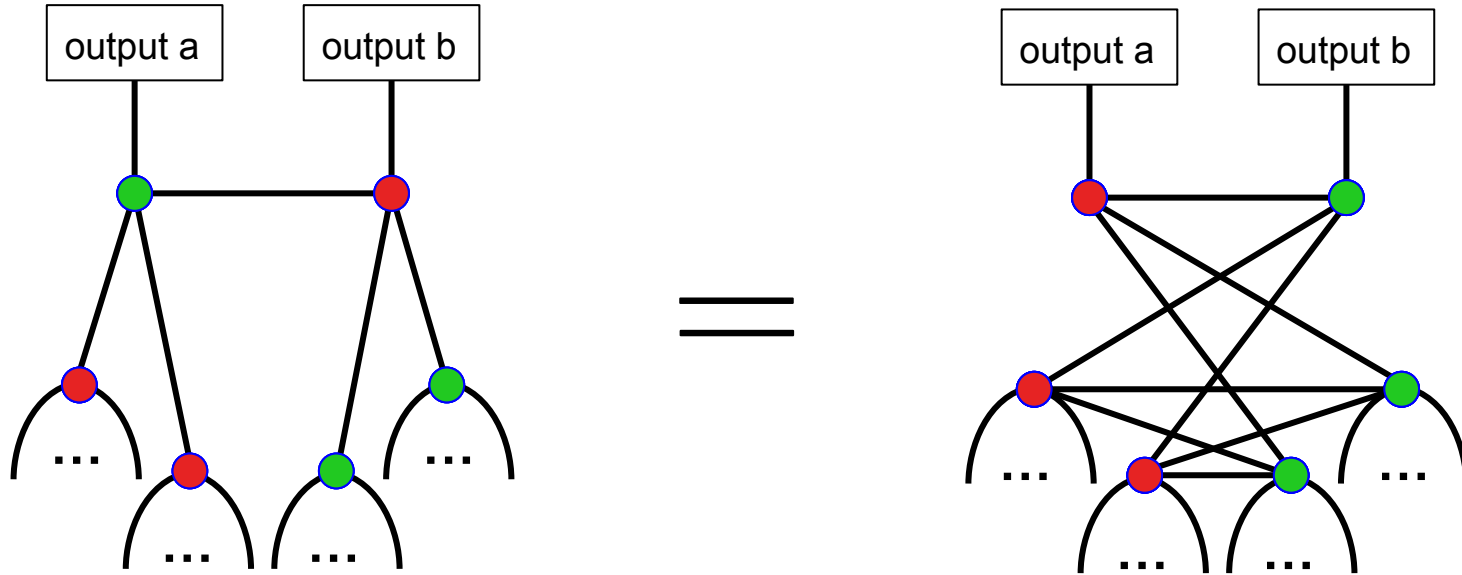
Example

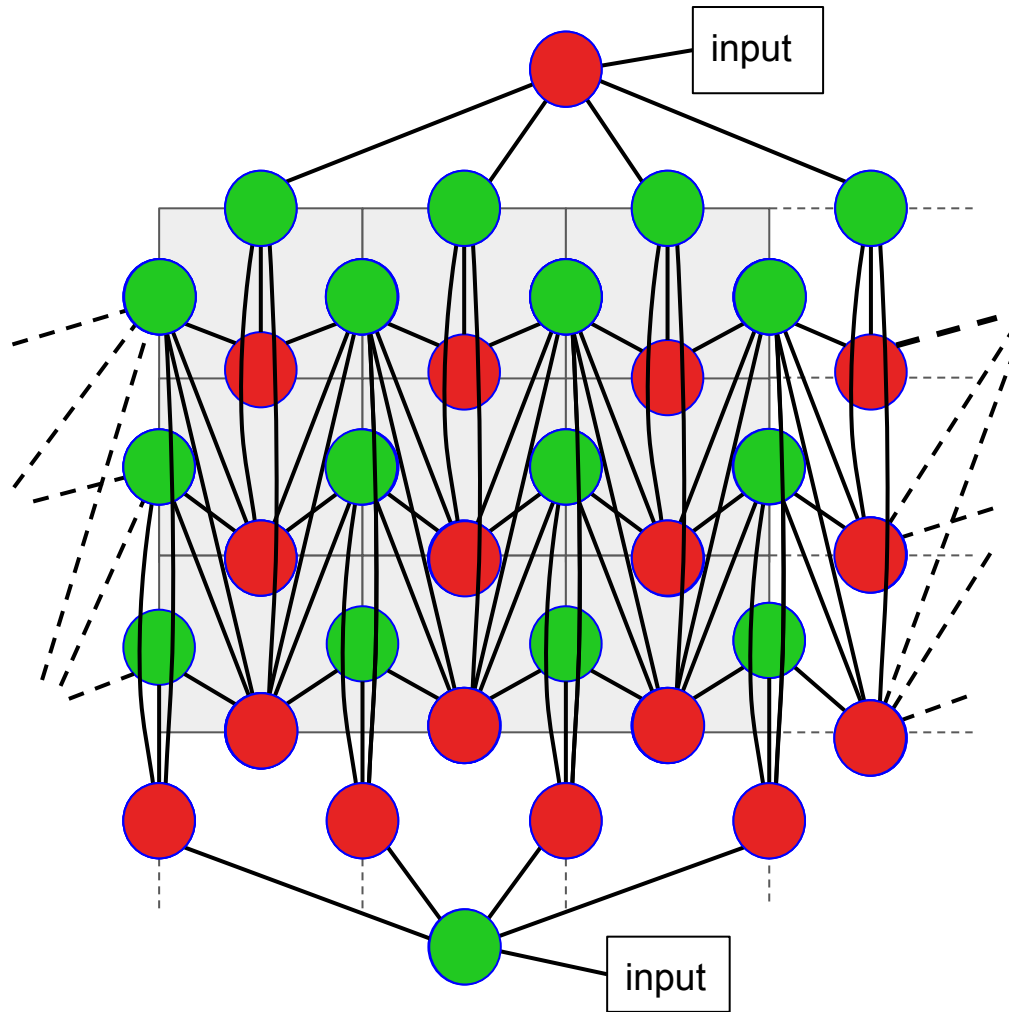


$=$



# Hadamard-slide rule rearranges output node colors

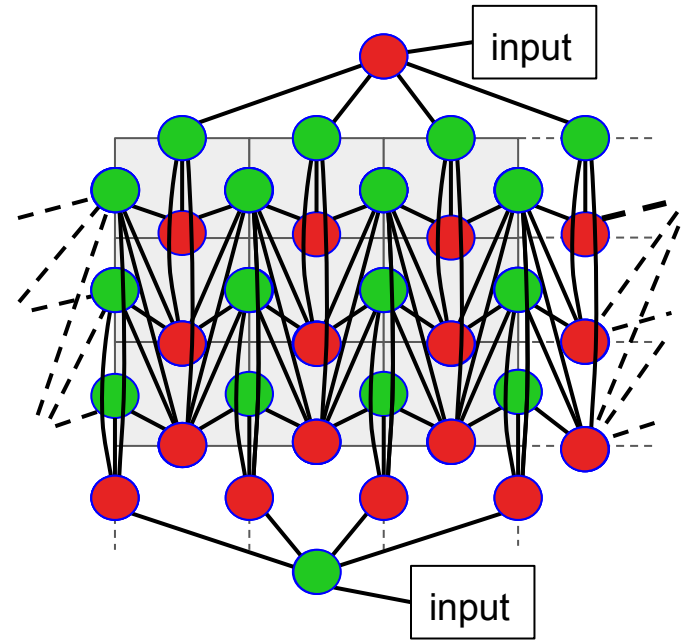




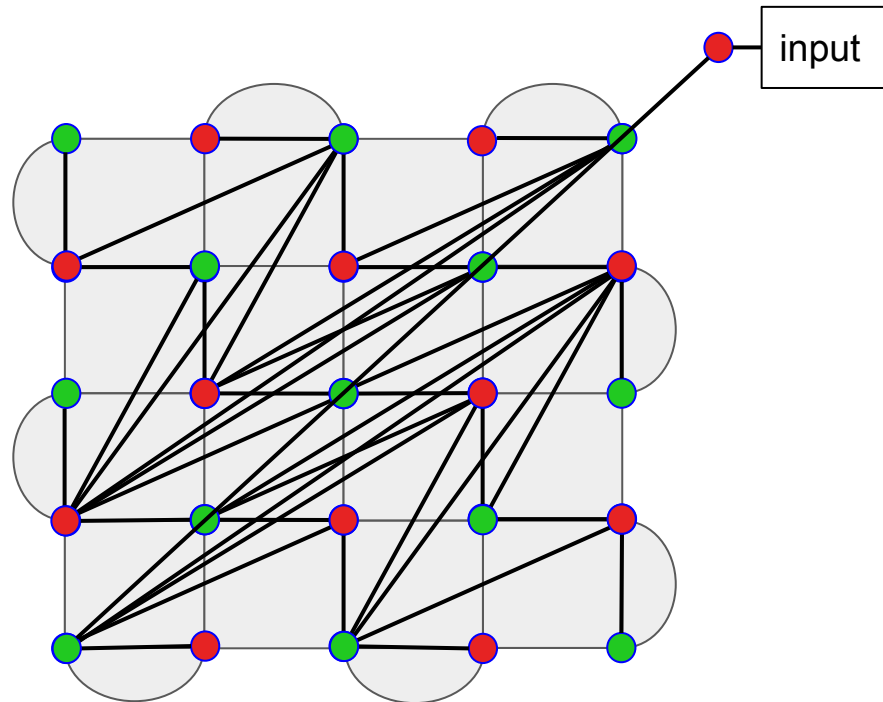
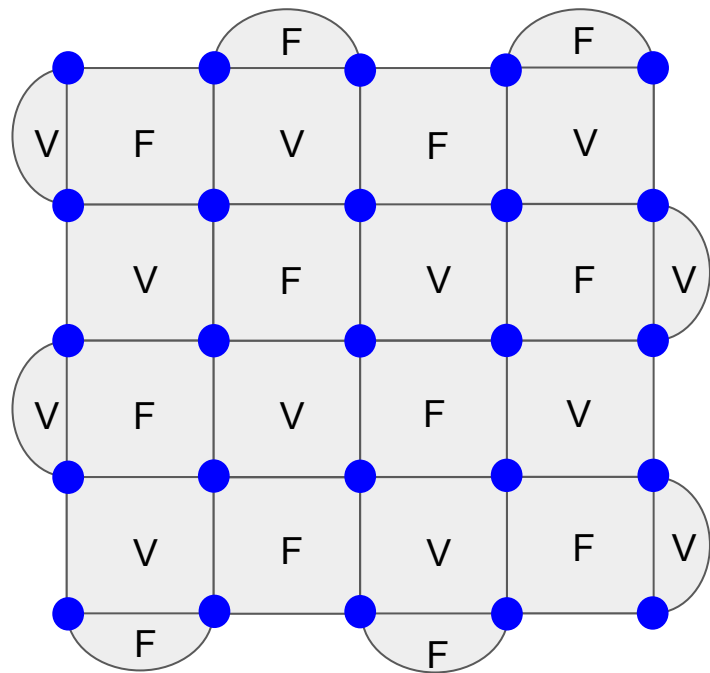
All nodes not labeled "input" are *output nodes*.

# Toric Code Diagram Properties

- Edges among outputs are *local* in horizontal direction
- Number of nodes is reduced from the Kissinger *normal form*
- No internal nodes; all nodes are *qubits*
- Generalizable to any  $m$  by  $n$
- We derived the general canonical form for Toric codes.



# Surface codes



# Applications

- Surface codes in lattice surgery
- Building large-scale quantum computers
  - Advances in:
    - Physical simulations
    - Algorithms
    - Finance

# Acknowledgements

- My mentor Andrey Boris Khesin
- The PRIMES program
- My family, friends, and teachers