Arithmetic of Semisubtractive Semidomains

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Background: Groups

$$(\mathbb{Z}, +) = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

A group G is a set under one binary operator such that:

- G is closed;
- the operator is associative;
  - $(a + b) + c = a + (b + c)$;
- there is an identity element;
- each element has an inverse.

Example

- $(\mathbb{Q} \setminus \{0\}, \cdot)$;
- $(\{1, -1, i, -i\}, \cdot)$.

Background: Monoids

$$(\mathbb{N}_0, +) = \{0, 1, 2, \ldots\}$$

A monoid M is a set under one binary operator such that:

- M is closed;
- the operator is associative;
  - $(a + b) + c = a + (b + c)$;
- there is an identity element;
- ~~each element has an inverse.~~

Example

- $(\mathbb{Z}, +)$;
- $(\mathbb{N}, \cdot)$.

Background: Integral Domains

$$(\mathbb{Z}, +, \cdot)$$

An integral domain D is a set under two binary operators such that:

- $(D, +)$ is a group;
- $(D \setminus \{0\}, \cdot)$ is a monoid;
- Multiplication distributes over addition;
- The only zero-divisor is 0.

An element $d \in D$ is a zero-divisor if there exists a nonzero element d' such that $dd' = 0$.

Example

- $(\mathbb{Z}[x], +, \cdot)$.

NOT an integral domain: $\mathbb{Z}/4\mathbb{Z} = \{0, 1, 2, 3\}$ ("integers mod 4").

Background: Semidomains

$$(\mathbb{N}_0, +, \cdot)$$

A semidomain S is a subset of an integral domain such that:

- $(S, +)$ is a monoid;
- $(S \setminus \{0\}, \cdot)$ is a monoid.

Example

- $(\mathbb{N}_0[x], +, \cdot)$;
- $(\mathbb{Z}[x], +, \cdot)$.

Every integral domain is a semidomain.

Background: Semisubtractive Semidomains

Given a semidomain S, we will define its group of differences, denoted by $\mathscr{G}(S)$. The object $\mathscr{G}(S)$ is also called the Grothendieck group of S.

$\mathscr{G}(S)$ consists of pairs of elements (a, b) for $a, b \in S$, representing the value $a - b$. We define $a - b$ to be equal to $c - d$ if $a + d = b + c$.

$\mathscr{G}(S)$ is not only a semidomain, but an integral domain. In fact, it is the least integral domain containing S. Thus, S is a subset of the integral domain $\mathscr{G}(S)$.

Examples

- $\mathscr{G}(\mathbb{N}_0) = \mathbb{Z}$;
- $\mathscr{G}(\mathbb{N}_0[x]) = \mathbb{Z}[x]$.

Background: Semisubtractive Semidomains

We say a semidomain S is semisubtractive if for all $a, b \in S$, either $a - b$ or $b - a$ is in S. More formally, there must exist some $x \in S$ such that $a + x = b$ or $b + x = a$.

Example

$\mathbb{N}_0$ is a semisubtractive semidomain. For $a, b \in \mathbb{N}_0$, $a - b \in \mathbb{N}_0$ if $a \geq b$ and $b - a \in \mathbb{N}_0$ if $b \geq a$. However, it is not an integral domain because none of its positive elements have additive inverses.

Example

$S = \mathbb{N}_0 + x\mathbb{Z}[x]$ also forms a semisubtractive semidomain. For polynomials $P, Q \in S$, at least one of $P - Q, Q - P$ is in S depending on which has a greater constant term. However, it is not an integral domain because 1, for example, does not have an additive inverse.

Background: Semisubtractive Semidomains

Equivalently, we may say that a semidomain S is semisubtractive if for all $s \in \mathscr{G}(S)$, either s or $-s$ is in S. Since all $s \in \mathscr{G}(S)$ can be written as $a - b$ for $a, b \in S$, this is equivalent to our previous example.

Example

Consider the semidomain S of integer polynomials whose lowest degree term is positive (in addition to 0).

The Grothendieck group of S will be $\mathbb{Z}[x]$.

Finally, this set is closed under multiplication and addition.

Motivation

Why should we care about these objects?

- Semisubtractive semidomains generalize the properties of integral domains.
- What properties are shared between integral domains and semisubtractive semidomains?
- In future research, we only have to prove that an object is a semisubtractive semidomain to understand its properties.
- Semisubtractive semidomains correspond closely to the natural numbers.

Factorization Properties

Let S be a semisubtractive semidomain.

- An element $u \in S$ is invertible if there exists $u' \in S$ such that $uu' = 1$.
- An element $a \in S$ is an atom if a cannot be expressed as a product of two non-invertible elements of S. The set of atoms of S is denoted $\mathscr{A}(S)$.
- S is atomic if every element of S can be expressed as a product of atoms.
- Two factorizations are considered the same if the atoms in the two factorizations only differ by invertible elements. (For example, in $\mathbb{Z}$, $14 = 2 \cdot 7$ and $14 = (-2)(-7)$ would be considered the same factorization, because -1 is invertible.)

Example

- $\mathbb{N}_0$;
- $\mathbb{N}_0[x]$.

Factorization Properties

There are several properties that an atomic semisubtractive semidomain can have that describe the factorizations of elements.

- Bounded Factorization
- Finite Factorization
- Half-Factorial
- Unique Factorization

Question

How do the factorization properties of S relate to the factorization properties of $\mathscr{G}(S)$?

Bounded Factorization

Definition

An atomic semisubtractive semidomain S is a bounded factorization semidomain (BFS) if for each $s \in S$, there are finitely many lengths that a factorization of s can have.

Example

- The Gaussian integers $\mathbb{Z}[i]$.

Theorem (Fox-Goel-Liao, 2023)

Let S be a semisubtractive semidomain. Then, S is a bounded factorization semidomain iff $\mathscr{G}(S)$ is a bounded factorization domain.

Finite Factorization

Definition

An atomic semisubtractive semidomain S is a finite factorization semidomain (FFS) if for each $s \in S$, there are finitely many factorizations of s.

Every FFS is a BFS.

Example

- $\mathbb{N}_0 + x^2\mathbb{N}_0[x]$;
- $\mathbb{N}_0 + x\mathbb{Z}[x]$.

Theorem (Fox-Goel-Liao, 2023)

Let S be a semisubtractive semidomain. Then, S is a finite factorization semidomain iff $\mathscr{G}(S)$ is a finite factorization domain.

Half-Factorial

Definition

An atomic semisubtractive semidomain S is a half-factorial semidomain if for each $s \in S$, there is one possible length of factorization of s.

Every HFS is a BFS.

Example

- $\mathbb{N}_0 + \mathbb{Z}\sqrt{2}$;
- $\sum_{p \in \mathbb{N}_0 + x\mathbb{Z}[x]} \mathbb{N}_0 y^p$.

Theorem (Fox-Goel-Liao, 2023)

Let S be a semisubtractive semidomain. Then, S is a half-factorial semidomain iff $\mathscr{G}(S)$ is a half-factorial domain and $\mathscr{A}(S) = S \cap \mathscr{A}(\mathscr{G}(S))$.

Unique Factorization

Definition

An atomic semisubtractive semidomain S is a unique factorization semidomain (UFS) if for each $s \in S$, there is one factorizations of s.

Every UFS is an HFS and FFS.

Example

- $\mathbb{N}_0$;
- $\mathbb{N}_0 + x\mathbb{Z}[x]$.

Theorem (Fox-Goel-Liao, 2023)

Let S be a semisubtractive semidomain. If S is a unique factorization semidomain, then $\mathscr{G}(S)$ is a unique factorization domain.

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Questions?