On the Winning and Losing Conditions of Schmidt's Games

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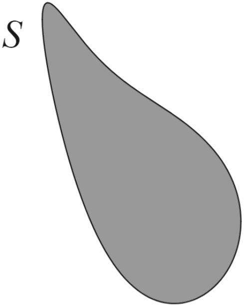
Let $0 < \alpha, \beta < 1$, and let $S \subset \mathbb{R}^n$. The game is played by two players: Alice and Bob. Bob starts first, and picks any ball B_0 with radius $r(B_0)$:



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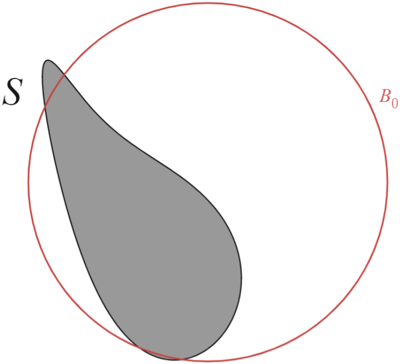
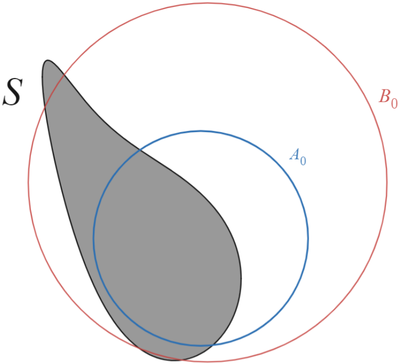
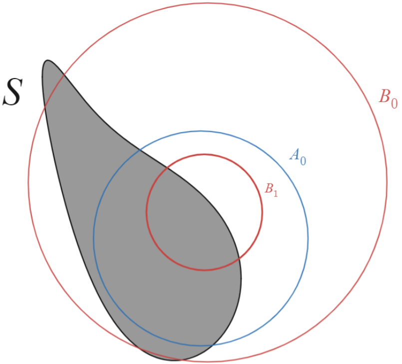


Then, Alice will play a ball A_0 such that $A_0 \subset B_0$ and $r(A_0) = \alpha r(B_0)$:

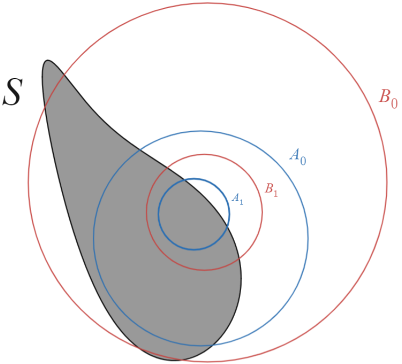


Then, Bob will play a ball B_1 such that $B_1 \subset A_0$ and $r(B_1) = \beta r(A_0)$:



Both players continue playing indefinitely, alternating balls, where

$$r(A_i) = \alpha r(B_i),\ \ r(B_{i+1}) = \beta r(A_i)\ \text{ for all } i = 0,\ 1,\ \ldots.$$



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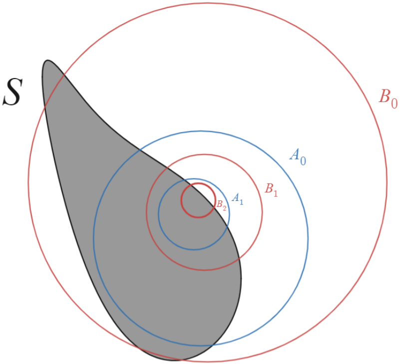
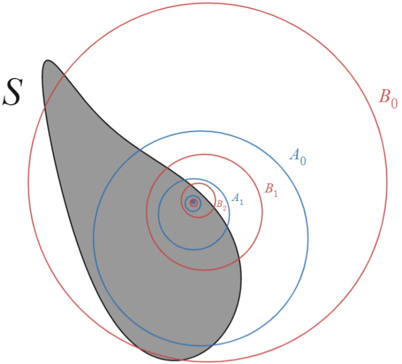


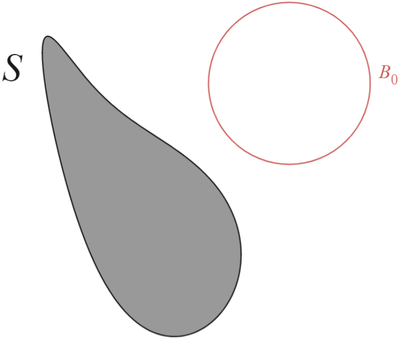
If the limit point

$$x = \bigcap_{i=0}^{\infty} A_i = \bigcap_{i=0}^{\infty} B_i$$

is in S, then Alice wins. If not, Bob wins.



We say S is (α, β)-winning if Alice is able to win no matter how Bob plays. Clearly, S in this example is not (α, β)-winning:



Schmidt's game in $\mathbb{R}$

Let $0 < \alpha, \beta < 1$. Suppose that two players Bob and Alice choose in turn a nested sequence of closed intervals in $\mathbb{R}$:

$$B_0 \supset A_0 \supset B_1 \supset \dots$$

with the property

$$|A_i| = \alpha |B_i|, \; |B_{i+1}| = \beta |A_i| \text{ for all } i = 0, \; 1, \; \dots .$$

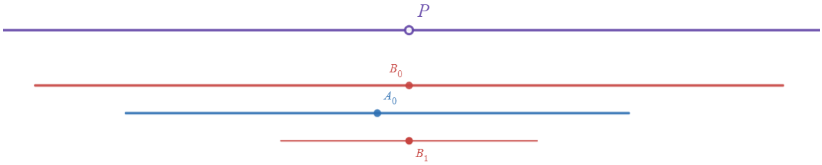
A set $S \subset \mathbb{R}$ is (α, β)-*winning* if Alice can pick intervals $\{A_i\}$ guaranteeing that the intersection

$$x = \bigcap_{i=0}^{\infty} A_i = \bigcap_{i=0}^{\infty} B_i$$

is in S no matter how Bob plays.

Lemma
Let $S = (-\infty, P) \cup (P, \infty)$. If $1 - 2\alpha + \alpha\beta \leq 0$, then S is (α, β)-losing.

Proof.
Bob selects B_0 centered at $P \subset S$. For all future turns, no matter what Alice plays, it is always possible for Bob to play such that B_i is centered at P. Clearly, the limit point is P. Therefore, Bob wins. □



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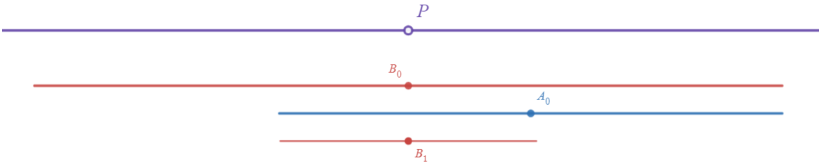

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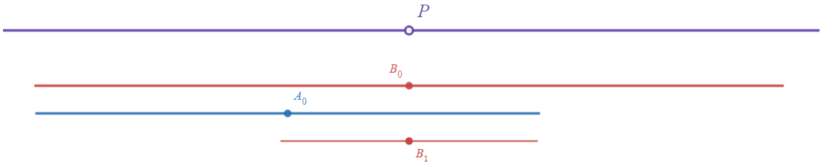


Definition

Denote by I the open unit square:

$$I := \{(\alpha, \beta) : 0 < \alpha, \beta < 1\} = (0, 1) \times (0, 1).$$

For any S, define the *Schmidt Diagram* $D(S)$ of S as the set of all pairs $(\alpha, \beta) \in I$ such that S is (α, β)-winning.

Definition

Let

$$\check{D} := \{(\alpha, \beta) \in I : 1 - 2\beta + \alpha\beta \leq 0\}$$

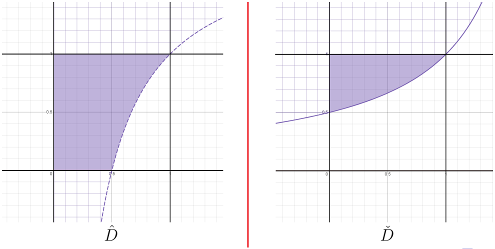
and

$$\hat{D} := \{(\alpha, \beta) \in I : 1 - 2\alpha + \alpha\beta > 0\}.$$

There are only four Schmidt Diagrams that are completely described: $\emptyset$, $\check{D}$, $\hat{D}$, and I.

Lemma

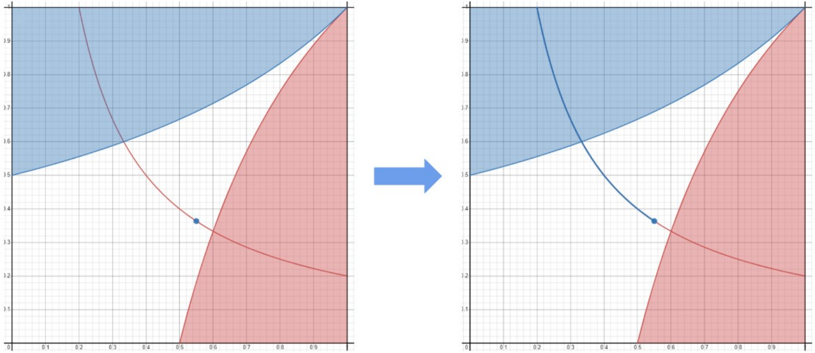
If S is dense and $S \neq \mathbb{R}$, then $\check{D} \subseteq D(S) \subseteq \hat{D}$.



Properties of Schmidt Diagrams

Lemma

If S is (α, β)-winning, $\alpha'\beta' = \alpha\beta$, and $\alpha' < \alpha$, then S is also (α', β')-winning.



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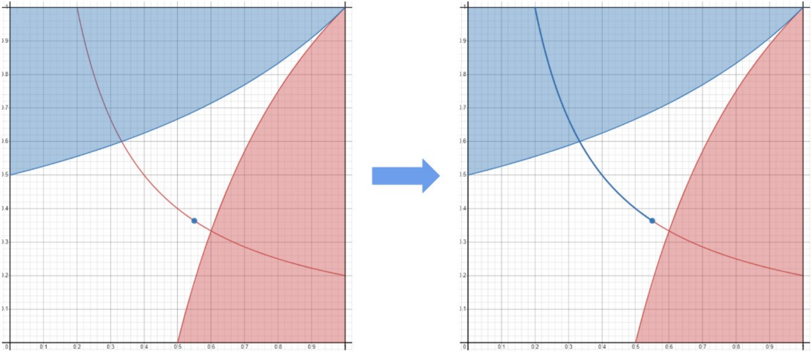

Lemma

If S is (α, β)-winning and $\alpha' < \alpha$, it does not follow that S is (α', β)-winning.

Diophantine Approximations deal with the approximation of real numbers using rational numbers.

Example

$\sqrt{2}$ can be approximated by the sequence of fractions

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \ldots$$

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Theorem

For any irrational number x, there exists infinitely many pairs of integers p, q such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}.$$

Theorem

For any irrational number x, there exists infinitely many pairs of integers p, k such that

$$\left|x - \frac{p}{2^k}\right| < \frac{1}{2^k}.$$

Let's consider

$$\text{2-BA} := \left\{x \in \mathbb{R} : \left|x - \frac{m}{2^n}\right| > \frac{c}{2^n} \text{ for some } c > 0 \text{ and all } m \in \mathbb{Z},\ n \in \mathbb{N}\right\}.$$

Theorem

Despite having zero Lebesgue measure, $D(\text{2-BA}) = \hat{D}$.

Variations

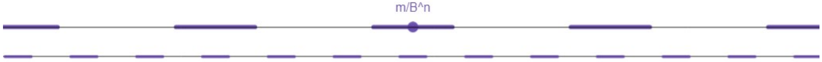
Define

$$2\text{-BA}(c,N) := \left\{ x \in \mathbb{R} : \left| x - \frac{m}{2^k} \right| > \frac{c}{2^k} \text{ for all } m \in \mathbb{Z},\ k \in \mathbb{N} \text{ s.t. } k > N \right\},$$

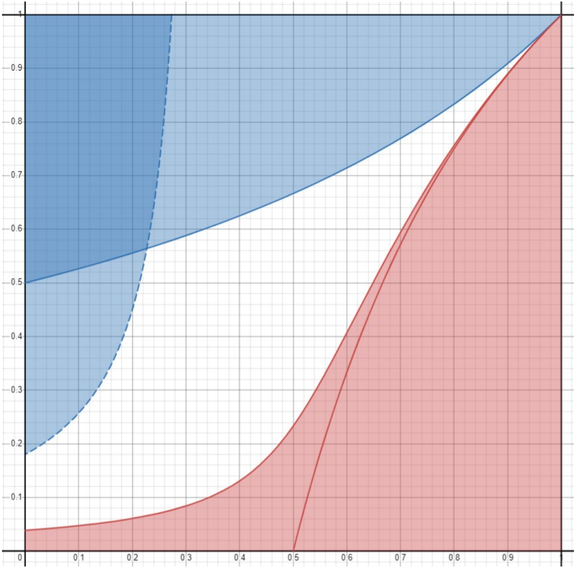
$$2\text{-BA}(c) := \bigcup_{N \in \mathbb{N}} 2\text{-BA}(c,N).$$

Its complement is equivalent to

$$2\text{-BA}(c)^c = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \left(\bigcup_{m \in \mathbb{Z}} \left[\frac{m}{2^n} - \frac{c}{2^n}, \frac{m}{2^n} + \frac{c}{2^n} \right] \right).$$



2-BA(c) bounds



Definition

Consider the base-2 expansions of the form $x = x_0.x_1x_2\cdots$ where x_0 is an integer and $x_i \in \{0,1\}$ are the digits in the base-2 expansion of x. We define

$$d^-(x,j) = \liminf_{k\to\infty} \frac{\#\{1 \le i \le k : x_i = j\}}{k}.$$

and the set

$$D_c^- = \{x \in \mathbb{R} : d^-(x,0) > c\}.$$

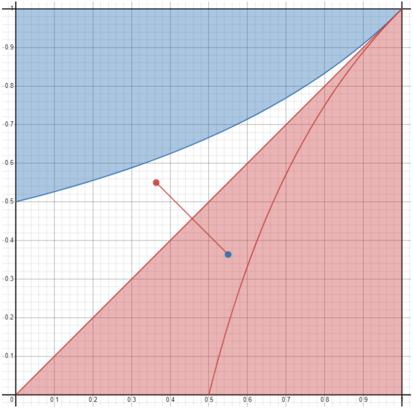
Example

If $x = 0.0101010101\ldots$, then $d^-(x,0) = \frac{1}{2}$.

Instead of focusing on a method to win, focus on overarching strategies as elements.

Theorem

The set $D^{-}_{1/2}$ is losing for $\alpha \geq \beta$.



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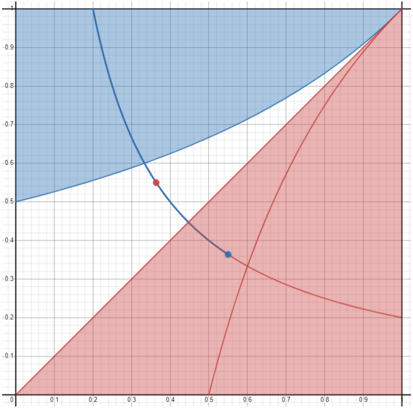

Conjecture

The set $D^-_{1/2}$ is winning for $\alpha < \beta$.

- Not trivially easy to prove, since the game is still inherently asymmetric: Bob picks his interval first.
- Furthermore, the $d^-(x, 0) = 1/2$ case makes things complicated.
- If proven true, this will produce a fifth completely described Schmidt Diagram.

Acknowledgements

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- P. Bohl, *Über ein in der Theorie der säkularen Störungen vorkommendes Problem, J. reine angew*, Math. (1909).
- Á. Farkas and J. Fraser and E. Nesharim and D. Simmons, *Schmidt's game on Hausdorff metric and function spaces: generic dimension of sets and images*, arXiv e-prints (2021), https://arxiv.org/pdf/1907.07394.pdf.
- S. Kalia and M. Zanger-Tishler, *On the Winning and Losing Parameters of Schmidt's Game*, PRIMES Preprint (2012), https://math.mit.edu/research/highschool/primes/materials/2012/Zanger-Tishler-Kalia.pdf.
- J. Nilsson, *The Fine Structure of Dyadically Badly Approximable Numbers*, arXiv e-prints (2010), https://arxiv.org/pdf/1002.4614.pdf.
- T. Persson and J. Schmeling, *Dyadic Diophantine Approximation and Katok's Horseshoe Approximation*, Acta Arithmetica **132** (2008), 3, 205–230.

References

- W. M. Schmidt, *On badly approximable numbers and certain games*, Trans. Amer. Math. Soc. **123** (1966), 178–199.
- B. Volkmann. Gewinnmengen, Arch. Math., **10** (1959), 235–240.

Thank you for your attention!