Random Constraint Satisfaction Problems: Coloring Hypergraphs and NAE-SAT

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- We want to see if our variables can satisfy those constraints.

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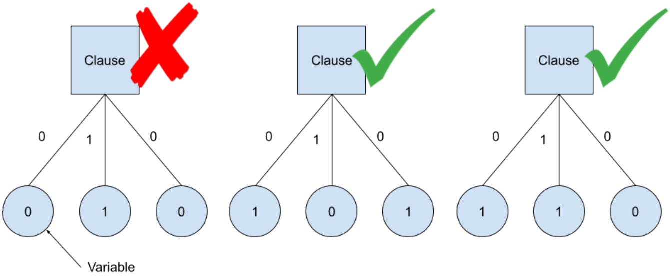

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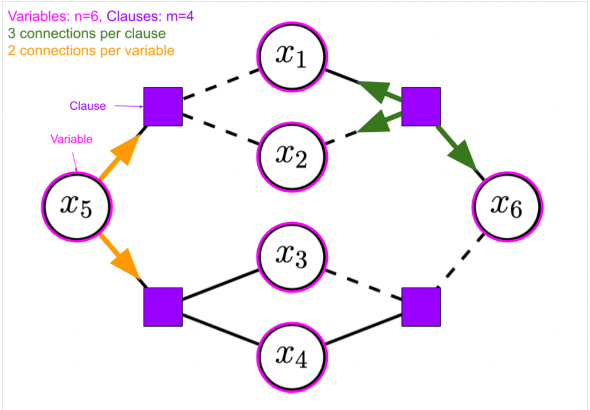
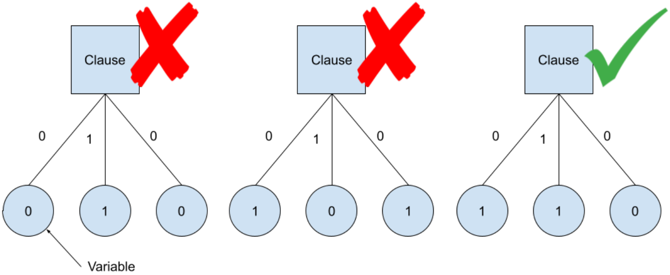
- m is total # of clauses, each clause imposed on k variables. n is total # of variables, d clauses imposed on each variable
- $d \cdot n = k \cdot m$. Why?



“Regular, and Not all Equals-SAT”

- Furthermore, we now say a clause is dissatisfied iff every one of its k variables matches its connection to clause OR every one of its k variables differs from its connection with clause



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- This means clauses and literals (recall literals are connection labels) are chosen randomly (so long as instance is d-regular)
- Intuitively, when there's a higher density of clauses (constraints), it's harder for variables to satisfy clauses.

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- Specifically, when α gets higher, it will pass a **satisfiability threshold,** before which probability of satisfiability always tends to one, and after which probability of satisfiability always tends to zero.

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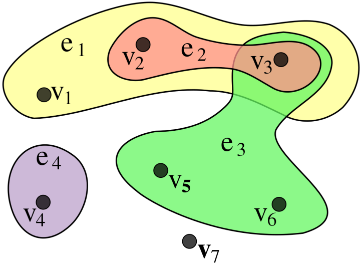
- **Hypergraph:** connections can involve more than two nodes.
- These connections are called **"hyperedges"**



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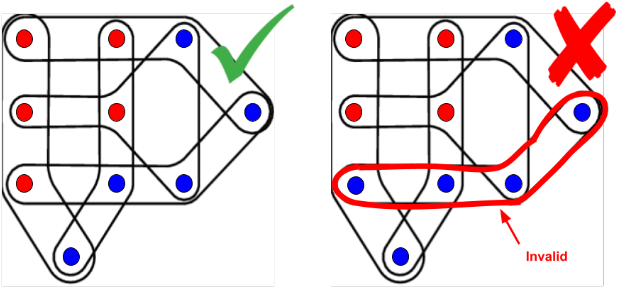
- Make every hyperedge consist of k nodes, each node part of d hyperedges ("d-regular"). [HY15]
- Can we assign colors from $\{\text{red}, \text{blue}\} \equiv \{0, 1\}$ to nodes so there's no **monochromatic** (same color) hyperedge?



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- Conjecture: same satisfiability threshold as the NAE-SAT?

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- Notice this is not the same as $(E[X])^2$.
- Observe $E[g(X)] = \sum_i g(x_i) \times p(x_i)$

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- (First Moment Method). For a non-negative, integer-valued random variable X, then

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- If X is counting something, then $X > 0$ shows existence.

Second moment method

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- (Second Moment Method). For a non-negative, integer-valued random variable X with finite variance, then

$$P(X > 0) \geq \frac{E[X]^2}{E[X^2]}.$$

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- We show the threshold also holds for the hypergraph model.
- Algebraically prove our upper bound is well-defined.

Acknowledgements

- Our mentor Dr. Youngtak Sohn
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- Dr. Tanya Khovanova
- Our parents

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