ON THE SPUM AND SUM-DIAMETER OF PATHS

ARYAN BORA, YUNSEO CHOI, AND LUCAS TANG

Abstract. In a sum graph, the vertices are labeled with distinct positive integers, and two vertices are adjacent if the sum of their labels is equal to the label of another vertex. The spum of a graph G is defined as the minimum difference between the largest and smallest labels of a sum graph that consists of G in union with a minimum number of isolated vertices. More recently, Li introduced the sum-diameter of a graph G, which modifies the definition of spum by removing the requirement that the number of isolated vertices must be minimal. In this paper, we settle conjectures by Singla, Tiwari, and Tripathi and a conjecture by Li by evaluating the spum and the sum-diameter of paths.

Keywords. Sum graph, Spum, Sum-diameter, Graph labeling, Path

1. Introduction

In 1990, Harary [2] defined the *sum graph* $G(V, E)$ *of* $L \subseteq \mathbb{N}$ to be given by $V = L$ and $(u, v) \in E$ if $u + v \in L$ (see for example, Figure 1). Not every graph G is a sum graph of some set L; for example, no connected G is a sum graph, because in a sum graph, the vertex with the largest label must be isolated. Yet, Harary showed [2] that any graph G in union with at least a *sum number* $\sigma(G)$ of isolated vertices is a sum graph of some set L.

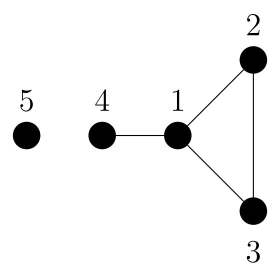

Figure 1. The sum graph of $\{1, 2, 3, 4, 5\}$

Shortly after, Goodell, Beveridge, Gallagher, Goodwin, Gyori, and Joseph [1] defined the *spum* $\text{spum}(G)$ *of a graph* G as the minimum difference between the largest and the smallest labels in L, for which $G \cup I_{\sigma(G)}$ is a sum graph of L. In the same paper, Goodell et al. [1] evaluated $\text{spum}(K_n)$. More recently, in 2021, Singla, Tiwari, and Tripathi [5] evaluated $\text{spum}(K_{1,n})$ and $\text{spum}(K_{n,n})$. Our first main result is that we settle a conjecture that was originally proposed by Singla, Tiwari, and Tripathi (see Conjecture 7.1 of [5]) and subsequently modified by Li (see Conjecture 3.4 of [4]) on $\text{spum}(P_n)$, where P_n is a path of n vertices.

Theorem 1.1. *For $n \geq 3$, it holds that*

$$\operatorname{spum}(P_n) = \begin{cases} 2n-3 & \text{if } 3 \leq n \leq 6 \\ 2n-2 & \text{if } n = 7 \\ 2n-1 & \text{if } n \geq 8 \text{ is even} \\ 2n+1 & \text{if } n \geq 9 \text{ is odd.} \end{cases}$$

In 1994, Harary [3] extended the notion of a sum graph; he defined the *integral sum graph* $G(V,E)$ *of* $L \subseteq \mathbb{Z}$ to be given by $V = L$ and $(u,v) \in E$ if $u + v \in L$. He then defined the *integral sum number* $\zeta(G)$ as the minimum number of vertices for which $G \cup I_{\zeta(G)}$ is an integral sum graph. In 2021, Singla et al. [5] extended the notion of spum; they defined the *integral spum* $\operatorname{ispum}(G)$ *of a graph* G as the minimum difference between the largest and the smallest labels in L, for which $G \cup I_{\zeta(G)}$ is an integral sum graph of L. In the same paper, Singla et al. [5] evaluated $\operatorname{ispum}(K_n)$, $\operatorname{ispum}(K_{1,n})$, and $\operatorname{ispum}(K_{n,n})$. Our next result is that we improve the best known lower bound on $\operatorname{ispum}(P_n)$.

Theorem 1.2. *For $n \geq 7$, it holds that*

$$2n-3 \leq \operatorname{ispum}(P_n) \leq \begin{cases} 2n-3 & \text{if } n \text{ is even} \\ \frac{5}{2}(n-3) & \text{if } n \text{ is odd.} \end{cases}$$

Last year, Li [4] introduced the more natural *sum diameter* $\operatorname{sd}(G)$ *of a graph* G as the minimum difference between the largest and the smallest labels of L, for which $G \cup I_m$ is a sum graph of L for *any* $m \geq \sigma(G)$.

Remark. Although a priori, it is not clear that adding more vertices reduces the range of its labels, there exist graphs G for which $\operatorname{sd}(G) < \operatorname{spum}(G)$. For example, while $\operatorname{spum}(P_8) = 15$ by Theorem 1.1, Figure 2 shows that $\operatorname{sd}(P_8) \leq 14$.

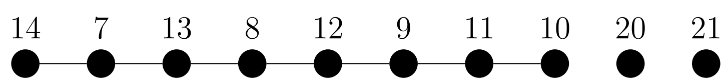

FIGURE 2. A sum graph that demonstrates $\operatorname{sd}(P_8) \leq 14$

Our next result is that we settle a conjecture by Li (see Conjecture 9.5 of [4]) on $\operatorname{sd}(P_n)$.

Theorem 1.3. *For $n \geq 3$, it holds that*

$$\operatorname{sd}(P_n) = \begin{cases} 2n-3 & \text{if } 3 \leq n \leq 6 \\ 2n-2 & \text{if } n \geq 7 \end{cases}.$$

Li [4] also introduced the *integral sum diameter* $\operatorname{isd}(G)$ *of a graph* G as the minimum difference between the largest and the smallest labels of L, for which $G \cup I_m$ is an integral sum graph of L for *any* $m \geq \zeta(G)$. Our last result is that we evaluate $\operatorname{isd}(P_n)$ for $n \geq 3$.

Theorem 1.4. *For $n \geq 3$, it holds that*

$$\operatorname{isd}(P_n) = \begin{cases} 2n-3 & \text{if } n = 3 \\ 2n-4 & \text{if } 4 \leq n \leq 7 \\ 2n-3 & \text{if } 8 \leq n \leq 9 \\ 2n-3 & \text{if } n \geq 10 \text{ is even} \\ 2n-2 & \text{if } n \geq 11 \text{ is odd.} \end{cases}$$

In Section 2, we establish preliminaries. In Section 3, we prove Theorem 1.1. In Section 4, we prove Theorems 1.2 and 1.4. In Section 5, we prove Theorem 1.3.

2. Preliminaries

For $L \subseteq \mathbb{Z}$, let $L - a = \{\ell - a | \ell \in L\}$, and $L + a = \{\ell + a | \ell \in L\}$. Next, let $\mathrm{range}(L) = \max(L) - \min(L)$. Furthermore, let $N_L(\ell) = \{a \in L \mid a \neq \ell \text{ and } \ell + a \in L\}$, and let $L_i \subseteq L$ be the i smallest numbers in L. For $L = \{a_1, a_2, \ldots, a_n\} \subseteq \mathbb{Z}$ such that $a_1 < a_2 < \cdots < a_n$, say that a_i and a_{i+1} are *consecutive* in L.

Let the maximal and minimal degree of a graph G be Δ_G and δ_G, respectively. We omit the subscript when it is clear which G is being referred to.

In [2] and [3] respectively, Harary evaluated $\sigma(P_n)$ and $\zeta(P_n)$.

Lemma 2.1 ([2])**.** *For $n \geq 1$, it holds that $\sigma(P_n) = 1$.*

Lemma 2.2 (Theorem 3.1 of [3])**.** *For $n \geq 1$, it holds that $\zeta(P_n) = 0$.*

Finally, let a component of G be a connected subgraph of G that is not a subset of a larger connected subgraph.

3. The Spum of Paths

For this section, let $G \cup I_{\sigma(G)}$ be a sum graph of L that satisfies $\mathrm{range}(L) = \mathrm{spum}(G)$. Let $S = \{a_1, a_2, \ldots, a_n\}$ be labels of the vertices of G such that $a_1 < a_2 < \cdots < a_n$. Next, let

$$\begin{aligned} T &= [a_1, a_n] \setminus S, \\ M &= (S \setminus [a_1, 2a_1]) - a_1, \text{ and} \\ J &= (S \setminus [a_1, 3a_1]) - 2a_1. \end{aligned}$$

We say that L is *tight* if $T \subseteq M$ and k*-tight* if $|T \setminus M| = k$. Let $m = a_1 - |S \cap [a_1, 2a_1]|$. Then, because $|S \cap [a_1, 2a_1]| = a_1 - m$ and $S \sqcup T = [a_1, a_n]$, it follows that

$$|T \cap [a_1, 2a_1]| = |[a_1, 2a_1]| - (a_1 - m) = m + 1. \tag{3.1}$$

Finally, let $\epsilon = 1$ if $(a_1, a_n) \in E$, and $\epsilon = 0$ otherwise.

3.1. The inequality $\mathrm{spum}(P_n) \geq 2n - 2$ for $n \geq 7$.

We first cite the best known bounds on $\mathrm{spum}(P_n)$.

Theorem 3.1 (Theorem 3.1 of [4])**.** *For $n \geq 7$, it holds that*

$$2n - 2 \leq \mathrm{spum}(P_n) \leq \begin{cases} 2n + 1 & \text{if } n \text{ is odd} \\ 2n - 1 & \text{if } n \text{ is even.} \end{cases}$$

We first generalize and correct an error in the original presentation of Claim 2 in [5].

Lemma 3.2. *If $G \cup I_i$ is a sum graph of L and $\{a_1, 2a_1\} \subseteq L$, then*

$$\mathrm{range}(L) \geq 3n - a_1 - 3 + m - 4\Delta + \delta + 2\epsilon.$$

Proof. First, because $|N(a_1)| \leq \Delta$, at most $\Delta - \epsilon$ pairs of labels in $S \setminus \{a_1\}$ differ by a_1. As $[a_1, 2a_1] \subseteq S \sqcup T = [a_1, a_n]$, by Equation (3.1), it holds that

$$|S \cap [2a_1 + 1, 3a_1]| \leq |T \cap [a_1 + 1, 2a_1]| + \Delta - \epsilon = \Delta + m + 1 - \epsilon. \tag{3.2}$$

Because $a_n \notin M$, we have that $S \cap M \subseteq N(a_1) \setminus a_n$, so

$$|S \cap M| \leq |N(a_{r+1})| - \epsilon \leq \Delta - \epsilon. \tag{3.3}$$

Likewise, because $M \cap J \subseteq N(a_1) \setminus a_n$, it follows that

$$|M \cap J| \leq |N(a_{r+1})| - \epsilon \leq \Delta - \epsilon. \tag{3.4}$$

Similarly, because $S \cap J \subseteq N(2a_1) \cup 2a_1$, it follows that

$$|S \cap J| \leq |N(2a_1)| + 1 \leq \Delta + 1. \tag{3.5}$$

Notice that because $|M| = |S \setminus [a_1, 2a_1]|$, it follows that

$$|M| = |S| - |S \cap [a_1, 2a_1]| = n - (a_1 - m). \tag{3.6}$$

Because $M \subseteq S \sqcup T = [a_1, a_n]$, from Equations (3.3) and (3.6),

$$|T \cap M| = |M| - |S \cap M| \geq n - (a_1 - m) - (\Delta - \epsilon). \tag{3.7}$$

Now, from Equation (3.2),

$$\begin{aligned} |J| &= |S| - |S \cap [a_1, 2a_1]| - |S \cap [2a_1 + 1, 3a_1]| \\ &\geq n - (a_1 - m) - (\Delta + m + 1 - \epsilon) \\ &= n - a_1 - 1 - \Delta + \epsilon. \end{aligned} \tag{3.8}$$

As $J \subseteq S \sqcup T = [a_1, a_n]$, Equations (3.5) and (3.8) imply

$$\begin{aligned} |T \cap J| = |J| - |S \cap J| & \\ &\geq n - a_1 - 1 - \Delta + \epsilon - (\Delta + 1) \\ &\geq n - a_1 - 2 - 2\Delta + \epsilon. \end{aligned} \tag{3.9}$$

Lastly, from Equations (3.4), (3.7), and (3.9),

$$\begin{aligned} |T| &\geq |T \cap M| + |T \cap J| - |T \cap M \cap J| \geq |T \cap M| + |T \cap J| - |M \cap J| \\ &= (n - a_1 + m - \Delta + \epsilon) + (n - a_1 - 2 - 2\Delta + \epsilon) - (\Delta - \epsilon) \\ &= 2n - 4\Delta + 3\epsilon - 2a_1 + m - 2. \end{aligned} \tag{3.10}$$

Because $|T| = (a_n - a_1 + 1) - n$, from Equation (3.10),

$$a_n \geq 3n - a_1 - 3 + m - 4\Delta + 3\epsilon. \tag{3.11}$$

Now, because $\max(N(a_n)) \geq a_1 + \delta - \epsilon$, from Equation (3.11),

$$\text{range}(L) \geq (a_n + a_1 + \delta - \epsilon) - a_1 \geq 3n - a_1 - 3 + m - 4\Delta + \delta + 2\epsilon. \tag{3.12}$$ □

Remark. The proof of Claim 2 in [5] claimed that $|S \cap J| \leq \Delta$, but it is possible that $|S \cap J| = \Delta + 1$ if $2a_1 \in S \cap J$.

By specifying $G = P_n$ in Lemma 3.2, we arrive at the following corollary.

Corollary 3.3. *If* $\{a_1, 2a_1\} \subseteq S$, *then* $\text{spum}(P_n) \geq 3n - a_1 - 11 + 2\epsilon$.

3.2. **The inequality** $\text{spum}(P_n) \geq 2n-1$ **for** $n \geq 8$**.** For the rest of Section 3, let $G = P_n \cup I_{\sigma(P_n)} = P_n \cup I_1$ from Lemma 2.1. In addition, let $\{a_{n+1}\} = L \setminus S$. Let $b_i = a_{i+1} - a_i$ for $1 \leq i \leq n$, and c_ℓ be the number of $1 \leq i \leq n$ such that $b_i = \ell$. We first generalize Lemma 2.5 by Li [4].

Lemma 3.4. *If L is k-tight, then* $2n - 3 + m + k + \epsilon \leq a_n \leq \text{range}(L)$*.*

Proof. By letting $G = P_n$ in Equation (3.7),

$$|T \cap M| \geq n - a_1 + m - 2 + \epsilon. \tag{3.13}$$

Now, because $|T \setminus M| = k$ and $|T| = (a_n - a_1 + 1) - n$, from Equation (3.13),

$$(a_n - a_1 + 1) - n = |T| = |T \cap M| + k \geq n - a_1 + m - 2 + \epsilon + k. \tag{3.14}$$

The statement of the lower bound of a_n now follows from rearranging Equation (3.14). In addition, because $a_{n+1} \geq a_n + a_1$, it follows that $a_n \leq a_{n+1} - a_1 = \text{range}(L)$. □

An immediate corollary of Lemma 3.4 is a lower bound for $\text{range}(L)$.

Corollary 3.5. *If L is k-tight, then* $\text{range}(L) \geq 2n - 2 + m + k$*.*

Proof. Because $\max(N(a_n)) \geq a_1 + 1 - \epsilon$, from Lemma 3.4,

$$\text{range}(L) \geq a_n + a_1 + 1 - \epsilon - a_1 \geq 2n - 2 + m + k. \tag{3.15}$$

□

Next, we give an upper bound on a_1 for $k-$tight L.

Lemma 3.6. *If L is k-tight, then $a_1 \leq k + 2$.*

Proof. Suppose otherwise. If $M \neq \emptyset$, then

$$\max(M) = a_n - a_1 \leq a_n - k - 3, \tag{3.16}$$

so regardless of whether or not $M = \emptyset$, it must hold that

$$[a_n - k - 2, a_n] \cap M = \emptyset. \tag{3.17}$$

Now, from Equation (3.17),

$$[a_n - k - 2, a_n] \cap T \subseteq T \setminus M. \tag{3.18}$$

Because $S \sqcup T = [a_1, a_n]$ and L is $k-$tight, it holds from Equation (3.18) that

$$\begin{aligned} |S \cap [a_n - k - 2, a_n]| &= |[a_n - k - 2, a_n]| - |[a_n - k - 2, a_n] \cap T| \\ &\geq |[a_n - k - 2, a_n]| - |T \setminus M| \\ &= k + 3 - k = 3. \end{aligned} \tag{3.19}$$

Now, because $\sigma(G) = 1$ and $a_1 \geq k + 3$, all vertices with labels in $S \cap [a_n - k - 2, a_n]$ have one neighbor. However, exactly two vertices in P_n have one neighbor. Thus, $a_1 \leq k + 2$. □

Next, we derive a upper bound for a_n. To do so, we define

$$\begin{aligned} X &= (S \setminus a_1) - a_1, \\ Y &= (S \setminus [a_1, a_2]) - a_2, \text{ and} \\ Z &= [1, a_n] \setminus S. \end{aligned}$$

In addition, let $\mu = |N(a_1) \setminus X| + |N(a_2) \setminus Y|$.

Lemma 3.7. *If $2a_1 \notin L$, then* $|Z \setminus X| + |Z \setminus Y| \leq 2a_n - 4n + 8 - \mu$*.*

Proof. First, because $|N(a_1)| \leq 2$ and $2a_1 \notin L$,

$$|X \cap S| \leq 2 - |N(a_1) \setminus X|. \tag{3.20}$$

Furthermore, because $|N(a_2)| \leq 2$ and $a_2 \in S \cap Y$,

$$|Y \cap S| \leq 3 - |N(a_2) \setminus Y|. \tag{3.21}$$

Because $X, Y \subseteq Z \sqcup S = [1, a_n]$, from Equations (3.20) and (3.21),

$$|X \cap Z| = |X| - |X \cap S| \geq (n-1) - (2 - |N(a_1) \setminus X|) = n - 3 + |N(a_1) \setminus X|, \tag{3.22}$$

and

$$|Y \cap Z| = |Y| - |Y \cap S| \geq (n-2) - (3 - |N(a_2) \setminus Y|) = n - 5 + |N(a_2) \setminus Y|. \tag{3.23}$$

Next, because $|Z| = a_n - n$, from Equations (3.22) and (3.23), it holds that

$$|Z \setminus X| = |Z| - |Z \cap X| \leq (a_n - n) - n + 3 - |N(a_1) \setminus X|, \tag{3.24}$$

and

$$|Z \setminus Y| = |Z| - |Z \cap Y| \leq (a_n - n) - n + 5 - |N(a_2) \setminus Y|. \tag{3.25}$$

Finally, the statement of the lemma follows from adding Equations (3.24) and (3.25). $\square$

Next, we use Lemma 3.7 to bound n from above when $a_1 + 1 = a_2$. In the following proof, for some $1 \leq i \leq j \leq n$, we say that $[a_i, a_j]$ is a *run* if $[a_i, a_j] \subseteq S$ and $\{a_i - 1, a_j + 1\} \cap S = \emptyset$. Note that $S = \bigsqcup_{i=1}^{t} R_i$, where R_i are runs.

Lemma 3.8. *If $2a_1 \notin S$ and $a_1 + 1 = a_2$, then $n \leq a_1(a_n - 2n + 7 - \mu) + 2 - \epsilon$.*

Proof. From Lemma 3.7,

$$\begin{aligned} 2a_n - 4n + 8 - \mu &\geq |Z \setminus X| + |Z \setminus Y| \\ &\geq |Z| - |X \cap Y| = |Z| - |Y| + |Y \setminus X|. \end{aligned} \tag{3.26}$$

Because $|Y| = n - 2$ and $|Z| = a_n - n$, it follows from Equation (3.26) that

$$\begin{aligned} |Y \setminus X| &\leq |Y| - |Z| + 2a_n - 4n + 8 - \mu \\ &= (n-2) - (a_n - n) + 2a_n - 4n + 8 - \mu \\ &= a_n - 2n + 6 - \mu. \end{aligned} \tag{3.27}$$

Note that $y \in Y \setminus X$ if and only if $y + a_2 = y + a_1 + 1 \in S$ and $y + a_1 \notin S$. Now, for all runs R_i, unless $a_1, a_2 \in R_i$, it holds that $\min(R_i) - a_2 \in Y \setminus X$, because by the definition of runs, $\min(R_i) - a_2 + a_1 = \min(R_i) - 1 \notin S$. Thus, from Equation (3.27),

$$t \leq |Y \setminus X| + 1 \leq a_n + 7 - \mu. \tag{3.28}$$

Next, because at most $2 - \epsilon$ pairs of labels in S differ by a_1 and $2a_1 \notin S$,

$$\sum_{i=1}^{t} \max(|R_i| - a_1, 0) \leq 2 - \epsilon, \tag{3.29}$$

from which it follows that

$$|S| = \sum_{i=1}^{t} |R_i| \leq t \cdot a_1 + 2 - \epsilon. \tag{3.30}$$

Therefore, from Equations (3.27) and (3.28), $|S| = n \leq a_1(a_n - 2n + 7 - \mu) + 2 - \epsilon$. $\square$

Next, we use the labels not in $S \cup (T \setminus M)$ to find labels in S.

Proposition 3.9. *If $t \notin S$ and $t \notin T \setminus M$, then $t + a_1 \in S$. In addition, if $t > 2a_1$ and $t - a_1 \notin T \setminus M$, then $t - a_1 \in S$.*

Proof. First, if $t+a_1 \notin S$, then $t \notin M$, from which $t \in T \setminus M$. Next, if $t > 2a_1$ and $t - a_1 \notin S$, then from $S \sqcup T = [a_1, a_n]$, it holds that $t - a_1 \in T$. Now, because $t \notin S$, it holds that $t - a_1 \notin M$ by the definition of M. Therefore, $t - a_1 \in T \setminus M$, a contradiction. □

Next, we show that if $\mathrm{spum}(P_n) = 2n - 2$ and $n \geq 13$, then $\{a_1, 2a_1\} \subseteq L$.

Lemma 3.10. *If $\mathrm{spum}(P_n) = 2n - 2$ for $n \geq 13$, then $\{a_1, 2a_1\} \subseteq L$.*

Proof. Suppose otherwise. From Corollary 3.5, $m = k = 0$. By Lemma 3.6, $a_1 \leq 2$. First, if $a_1 = 2$, then $|S \cap [2, 4]| = 2$. Thus, because $2a_1 = 4 \notin S$, we have $a_2 = 3$. Then, because $a_n \leq \mathrm{spum}(P_n) = 2n - 2$, by Lemma 3.8, $n \leq 12$. Thus, $a_1 = 1$.

Now, as $m = 0$, it holds that $|S \cap [1, 2]| = 1$, so $2 \notin S$. As $k = 0$, it holds that $T \setminus M = \emptyset$, so by setting $t = 2$ in Proposition 3.9, $a_2 = 3$. Next, from Lemma 3.4, $2n - 3 \leq a_n \leq 2n - 2$. However, if $a_n = 2n - 3$, then $N(2n - 3) = \emptyset$ as $2 \notin S$. Thus, $a_n = 2n - 2$.

Now, if $2n - 3 \in S$, then because $2 \notin S$, the vertices $\{2n - 2, 1, 2n - 3\}$ are a component of G, so $2n - 3 \notin S$. Because $k = 0$, by setting $t = 2n - 3$ in Proposition 3.9, it holds that $2n-4 \in S$. Now, if 4 or $2n-5$ is in L, then $\{1, 3, 4, 2n-5, 2n-4, 2n-2\} \cap L$ is a component of G, so $\{4, 2n - 5\} \subseteq T$. Now, by setting $t = 4$ (resp. $2n - 5$) in Proposition 3.9, we have $5 \in S$ (resp. $2n - 6 \in S$). Because $\{2, 4, 2n - 2, 2n - 4\} \subseteq T$, it holds that $|N(2n - 6)| = 1$. However, the vertices $2n - 2$ and $2n - 4$ also have 1 neighbor. Thus, $\{a_1, 2a_1\} \subseteq L$. □

Now, from Lemma 3.10, we show that $\mathrm{spum}(P_n) \geq 2n - 1$.

Theorem 3.11. *If $n \geq 8$, then $\mathrm{spum}(P_n) \geq 2n - 1$.*

Proof. Suppose otherwise. First, from Theorem 3.1, $\mathrm{spum}(P_n) = 2n-2$. Then, by computer search in Li [4] for odd $8 \leq n \leq 12$, it follows that $n \geq 13$. Now, as $m \geq -1$, from Corollary 3.5, $k \leq 1$. By Lemma 3.6, $a_1 \leq 3$. However, from Lemma 3.10, $2a_1 \in S$. Thus, $2n - 2 = \mathrm{range}(L) \geq 3n - a_1 - 11$ by Corollary 3.3, which implies $n \leq a_1 + 9 \leq 12$. □

3.3. **The inequality $\mathrm{spum}(P_n) \geq 2n$ for odd $n \geq 9$.** We now show that $\mathrm{spum}(P_n) \geq 2n$ for odd $n \geq 9$. First, assign $\varphi : [n] \to \{0, 1\}$ such that

$$\varphi(i) = \begin{cases} 1 & \text{if } a_i \geq 8 \text{ and } b_{i-1} = b_{i-2} = b_{i-3} = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Next, we define for labels a_i in L

$$\mathrm{st}(a_i) = a_i - 2 \sum_{j=1}^{i-1} \varphi(j).$$

Note that $\mathrm{st}(a_i)$ is weakly increasing in i. Next, we define the set $\mathrm{st}(L)$

$$\mathrm{st}(L) = \{\mathrm{st}(a_i) | \varphi(i) = 0\}.$$

Note that each $\mathrm{st}(a_i)$ for i such that $\varphi(i) = 0$ are distinct. The following proposition follows immediately from the definitions.

Proposition 3.12. *If $\mathrm{st}(a_i) < \mathrm{st}(a_j)$ are consecutive in $\mathrm{st}(L)$ for $i \geq 2$, then $\mathrm{st}(a_j) - \mathrm{st}(a_i) = b_{j-1}$.*

Proof. If $j = i+1$, it holds that $\mathrm{st}(a_j) - \mathrm{st}(a_i) = a_{i+1} - a_i - 2\varphi(i) = b_i$. Otherwise, if $j \neq i+1$, then because $\mathrm{st}(a_\ell) \notin \mathrm{st}(L)$ and $\varphi(\ell) = 1$ for $i \leq \ell - 1 \leq j$, we have $b_\ell = 2$ for $i-2 \leq \ell \leq j-2$. In addition, because $\varphi(i+1) = 1$ and $\varphi(i) = 0$, it follows that $a_{i-3} \neq 2$. Thus, $\varphi(i-1) = 0$, and $\mathrm{st}(a_j) - \mathrm{st}(a_i) = b_{j-1} + 2 \cdot (j-i) - 2\sum_{\ell=i-1}^{j-1} \varphi(\ell) = b_{j-1}$. ☐

We now show that number of pairs of labels that differ by 3 in L is at least that of $\mathrm{st}(L)$.

Proposition 3.13. $|N_L(3)| \geq |N_{\mathrm{st}(L)}(3)|$.

Proof. We show that if $\mathrm{st}(a_i) - 3 \in N_{\mathrm{st}(L)}(3)$, then $a_i - 3 \in N_L(3)$. Suppose that there exists some a_j such that $\mathrm{st}(a_i) = \mathrm{st}(a_j) + 3$. First, from Proposition 3.12, if $[\mathrm{st}(a_j)+1, \mathrm{st}(a_j)+2] \cap \mathrm{st}(L) = \emptyset$, then $b_{i-1} = 3$. Thus, $a_i - 3 \in N_L(3)$. Next, if $[\mathrm{st}(a_j)+1, \mathrm{st}(a_j)+2] \cap \mathrm{st}(L) = \mathrm{st}(a_j) + 1 = \mathrm{st}(a_\nu)$ with $\varphi(\nu) = 0$, then $b_{i-1} = 2$ and $b_{\nu-1} = 1$ from Proposition 3.12. Now, because $b_{\nu-1} = 1$, it holds that if $i \neq \nu + 1$, then $\varphi(\nu+1) = 1$. Thus, $i = \nu + 1$, so $a_{\nu-1} = a_j - 3$.

Now, suppose $[\mathrm{st}(a_j)+1, \mathrm{st}(a_j)+2] \cap \mathrm{st}(L) = \mathrm{st}(a_j)+2 = \mathrm{st}(a_\nu)$ with $\varphi(\nu) = 0$. Then, from Proposition 3.12, $b_{i-1} = 1$ and $b_{\nu-1} = 2$. If $i = \nu+1$, then either $a_{\nu-1} = a_i - 3$ or $b_{i-2} = 2$ because $\varphi(i-1) = 1$. Thus, $a_{i-2} = a_j - 3$. Finally, suppose that $[\mathrm{st}(a_j)+1, \mathrm{st}(a_j)+2] \subseteq \mathrm{st}(L)$. If $\mathrm{st}(a_j)+1 = \mathrm{st}(a_\nu)$, and $\mathrm{st}(a_j)+2 = \mathrm{st}(a_\ell)$ with $\varphi(\nu) = \varphi(\ell) = 0$, then from Proposition 3.12, $b_{i-1} = b_{\ell-1} = b_{\nu-1} = 1$. If $i \neq \ell+1$, then $\varphi(\ell+1) = 1$ because $b_{\ell-1} = 1$. Thus, $i = \ell + 1$. Similarly, $\ell = \nu+1$. Thus, because $b_{i-1} = b_{\ell-1} = b_{\nu-1} = 1$, we have $a_i = a_{\nu-1} + 3$. ☐

Next, we prove that if $\mathrm{range}(L) = 2n-1$ and $a_1 = 1$, then $2 \in S$.

Lemma 3.14. *If* $\mathrm{range}(L) = 2n-1$ *for odd* $n \geq 17$ *and* $a_1 = 1$*, then* $[1,2] \subseteq S$.

Proof. Suppose otherwise. Then $m = 0$. Now, from Corollary 3.5, $k = |T \setminus M| \leq 1$. If $a_2 \geq 5$, then $\{2,3,4\} \subseteq T$, which implies that $\{2,3\} \subseteq T \setminus M$. It follows that $a_2 \in \{3,4\}$. Suppose that $a_2 = 4$. Then $2 \in T \setminus M$, so $k = 1$. From Lemma 3.4, $a_n \in \{2n-2, 2n-1\}$. Because $2 \in T$ and $a_{n+1} = 2n - 1 + a_1 = 2n$, if $a_n = 2n-2$, then $N(2n-2) = \emptyset$. Thus, $a_n = 2n-1$. Now, as $2 \notin S$ and $|N(1)| \leq 2$, if $2n-2 \in S$, then the labels $\{2n-1, 1, 2n-2\}$ form a component. Therefore, $2n-2 \in T$. By setting $t = 2n-2$ in Proposition 3.9, $2n-3 \in S$. However, if $\{2,3,2n-2\} \subseteq T$, then $N(2n-3) = \emptyset$. Thus, $a_2 = 3$.

Now, from Lemma 3.4, $a_n \geq 2n-3$. Because $2 \in T$ and $a_{n+1} = \mathrm{range}(L) + a_1 = 2n$, if $a_n = 2n-2$, then $N(2n-2) = \emptyset$. Thus, $a_n \in \{2n-3, 2n-1\}$. Furthermore, if $b_i \geq 3$, then $[a_i+1, a_i+2] \subseteq T$ and $a_i + 1 \in T \setminus M$. Therefore, $b_i \geq 3$ for at most one value of $i \in [1, n-1]$. In addition, because $\sum_{i=1}^{n} b_i = \mathrm{range}(L) = 2n-1$, it follows that $c_1 = 2$, $c_2 = n-3$, and $c_3 = 1$. Therefore, for $n \geq 10$, there exists a sequence $b_i = b_{i+1} = b_{i+2} = 2$ with $a_{i+2} \geq 8$. Thus, $|\mathrm{st}(L)| \leq 9$. From Proposition 3.13, we must have $N_{\mathrm{st}(L)}(3) \leq 2$. In addition, if $\{\max(\mathrm{st}(L))-2, \max(\mathrm{st}(L))-1\} \subseteq \mathrm{st}(L)$, then $\{1, \max(L)-2, \max(L)-1\} \subseteq L$ is a component. Thus, $\{\max(\mathrm{st}(L))-2, \max(\mathrm{st}(L))-1\} \not\subseteq \mathrm{st}(L)$. Now, Table 1 shows the exhaustive list of the possible $\mathrm{st}(L)$ found by computer search given the constraints

(1) $|\mathrm{st}(L)| \leq 9$,
(2) $N_{\mathrm{st}(L)}(3) \leq 2$, and
(3) $\{\max(\mathrm{st}(L))-2, \max(\mathrm{st}(L))-1\} \not\subseteq \mathrm{st}(L)$

and why each is not a sum graph labeling of P_n. Therefore, $[1,2] \subseteq S$. ☐

Now, we prove the analog of Lemma 3.14 for $a_1 \neq 1$.

st(L)	Why P_n is not a sum graph of L
$\{1,3,5,6,7,9,11,14\}$	$\{1,5,6\}$ is a cycle.
$\{1,3,6,7,9,11,12\}$	$N(2n-5)=\emptyset$
$\{1,3,4,6,8,11,12\}$	$\{6,8,10\}\subseteq N(4)$ for $n\geq 9$.
$\{1,3,5,6,9,11,13,14\}$	$\{2n-1,1,5,2n-5\}$ is a component.
$\{1,3,5,6,8,10,13,14\}$	$\{1,3,2n-6\}\subseteq N(5)$
$\{1,3,5,7,8,9,12\}$	$\lvert N(n)\rvert=\lvert N(n-2)\rvert=\lvert N(2n-1)\rvert=1$ for odd n.
$\{1,3,5,7,8,11,12\}$	$\lvert N(n)\rvert=\lvert N(n-2)\rvert=\lvert N(2n-1)\rvert=1$ for odd n.

Table 1. Remaining possible st(L) in the proof of Lemma 3.14

Lemma 3.15. *If* range$(L)=2n-1$ *for odd* $n\geq 17$ *and* $a_1\neq 1$*, then* $2a_1\in S$*.*

Proof. Suppose otherwise. Then from Corollary 3.5, $k\leq 1$. Thus, $a_1\in\{2,3\}$. First, suppose that $a_1=3$. Then, $1\leq |[a_n-2,a_n-1]\cup(T\setminus M)|\leq k$ by the definition of k. Thus, $m=0$ from Corollary 3.5, and so $[3,5]\subseteq S$. Because $a_n\geq 2n-2$ from Lemma 3.4 and $a_{n+1}=\text{range}(L)+3=2n+2$, it follows that $N(a_n)\cap\{3,4\}\neq\emptyset$, so $\mu\geq 1$. If $a_n=2n-1$, then $\epsilon=1$, and from Lemma 3.8, $n\leq 16$. Thus, $a_n=2n-2$, so from Lemma 3.8, $n\leq 14$.

Next, suppose that $a_1=2$. First, if $m=1$, then because $\{3,4\}\subseteq T$, $\max(L)=2n+1$ and $N(a_n)\neq\emptyset$, from Lemma 3.4, $a_n=2n-1$. Now, because $N(2n-2)=\emptyset$, then $2n-2\in T\setminus M$. Now, from Corollary 3.5, $k=0$. Thus, $m=0$ and $[2,3]\subseteq S$, so $n\leq 14$ from Lemma 3.8. □

Now, from Lemma 3.15, we show that spum$(P_n)\geq 2n$ for odd n.

Theorem 3.16. *For odd* $n\geq 9$*, it holds that* spum$(P_n)\geq 2n$*.*

Proof. Suppose otherwise. First, from Theorem 3.11, spum$(P_n)=2n-1$. From computer search by Li [4], $n\geq 17$. Now, because $m\geq -1$, from Corollary 3.5, $k\leq 2$. Then, from Lemma 3.6, $a_1\leq 4$. Furthermore, from Lemmas 3.14 and 3.15, it holds that $2a_1\in S$, and thus from Corollary 3.3, $2n-1=\text{range}(L)\geq 3n-a_1-11$, which implies that $n\leq a_1+10\leq 14$. □

3.4. **The inequality** spum$(P_n)\geq 2n+1$ **for odd** $n\geq 9$**.** We first show that if range$(L)=2n$ and $a_1=1$, then $a_2\leq 4$.

Lemma 3.17. *If* range$(L)=2n$ *for odd* $n\geq 17$*, and* $a_1=1$*, then* $a_2\leq 4$*,*

Proof. Suppose otherwise. Then $m=0$. From Corollary 3.5, $k=|T\setminus M|\leq 2$. Now, if $a_2\geq 6$, then $\{2,3,4,5\}\subseteq T$, and thus, $\{2,3,4\}\subseteq T\setminus M$. Therefore, $a_2=5$.

As a result, $\{2,3\}\subseteq T\setminus M$, so $k=2$. From Lemma 3.4, $a_n\geq 2n-1$. If $a_n=2n-1$, then $N(2n-1)=\emptyset$, because $2\in T$ and $a_{n+1}=\text{range}(L)+a_1=2n+1$. Thus, $a_n=2n$. Now, if $2n-1\in S$, then because $1\in S$ and $2\notin S$ and $|N(1)|\leq 2$, the vertices with labels $\{2n,1,2n-1\}$ form a component. Therefore, $2n-1\in T$. By setting $t=2n-1$ in Proposition 3.9, $2n-2\in S$. However, because $\{2,3,2n-1\}\subseteq T$, it holds that $N(2n-2)=\emptyset$. □

We now show the analog of Lemma 3.17 for $a_2=4$.

Lemma 3.18. *If* range$(L)=2n$ *for odd* $n\geq 17$ *and* $a_1=1$*, then* $a_2\neq 4$*.*

Proof. Suppose otherwise. Then, $2\in T\setminus M$, so $k\geq 1$. Now, from Lemma 3.4, $a_n\geq 2n-2$. If $a_n\in\{2n-2,2n-1\}$, then $N(a_n)=\emptyset$, because $\{2,3\}\subseteq T$ and $a_{n+1}=2n+1$. Thus, $a_n=2n$.

First, if $2n-1 \in S$, then because $1 \in S$ and $2 \notin S$ and $|N(1)| \leq 2$, the vertices with labels $\{2n, 1, 2n-1\}$ form a component. Therefore, $2n-1 \in T$. Furthermore, if $2n-2 \in S$, then because $\{2, 3, 2n-1\} \subseteq T$, we have $N(2n-2) = \emptyset$, so $2n-2 \in T$. Then, from Corollary 3.5, $k \leq 2$, so $\{2n-2, 2\} = T \setminus M$. Then, by setting $t = 2n-2$ in Proposition 3.9, $2n-3 \in S$.

Now, if $\{5, 2n-4\} \cap L \neq \emptyset$, then the vertices with labels in $\{1, 4, 5, 2n-4, 2n-3, 2n\} \cap S$ form a component because $|N(1)|, |N(4)| \leq 2$. Thus, because $n \geq 17$, it must hold that $\{5, 2n-4\} \cap S = \emptyset$. Now, setting $t = 5$ (resp. $t = 2n-4$) in Proposition 3.9, we have $6 \in S$ (resp. $2n-5 \in S$). But then, because $\{2, 3, 5, 2n-4, 2n-2, 2n-1\} \subseteq T$, it holds that $|N(2n-5)| = 1$. Therefore, the vertices with labels $2n, 2n-3$, and $2n-5$ all have degree 1, a contradiction to $G = P_n$. ☐

We now show the analog of Lemma 3.17 for $a_2 = 3$.

Lemma 3.19. *If* $\text{range}(L) = 2n$ *for odd* $n \geq 17$*, and* $a_1 = 1$*, then* $a_2 \neq 3$*.*

Proof. Suppose otherwise. Then from Lemma 3.4, $a_n \geq 2n-3$. First, because $|N(1)| \leq 2$, then $c_1 \leq 2$. Then because $\sum_{i=1}^{n} b_i = \text{range}(L) = 2n$, either

- $c_2 = n$,
- $c_1 = c_3 = 2$, and $c_2 = n-4$,
- $c_1 = 2$, $c_4 = 1$, and $c_2 = n-3$, or
- $c_1 = 1$, $c_2 = n-2$, and $c_3 = 1$.

First, if $c_2 = n$, then labels in L are odd, so $G = I_n \neq P_n$. Now, because $2 \in T$, we have $b_n \neq 2$ and $b_1 \neq 2$, otherwise $N(a_n) = \emptyset$ and $a_2 = 2$, respectively. Next, we show that if $c_1 = 2$, $c_4 = 1$, and $c_2 = n-3$, and $b_n = 1$, then $b_{n-1} = 4$. Suppose otherwise.

First, $b_{n-1} \neq 1$, otherwise $\{2n, 1, 2n-1\}$ is a component. In addition, if $b_{n-1} = 2$, then $\{4, 2n-3\} \cap L = \emptyset$, otherwise $\{1, 3, 4, 2n-3, 2n-2, 2n\} \cap S$ form a component, because $2 \in T$. In addition, $2n-4 \notin S$, because otherwise $\max(|N(2n-4)|, |N(2n)|, |N(2n-2)|) \leq 1$. Thus, $\{2n-4, 2n-3\} \subseteq T$, and it follows that $b_{n-2} = 4$ and $a_{n-2} = 2n-6$. Because $\max(|N(2n)|, |N(2n-2)|) \leq 1$, it holds that $|N(2n-6)| = 2$, and $\{6, 7\} \subseteq S$. However, $a_2 = 3$, $\{6, 7\} \subseteq S$, $c_3 = 0$, and $c_1 = 1$. Therefore, $b_{n-1} = 4$.

Therefore, for $n \geq 14$, there exists i such that $b_i = b_{i+1} = b_{i+2} = 2$ with $a_{i+3} \geq 8$. Thus, $|\text{st}(L)| \leq 13$. From Proposition 3.13, $N_{\text{st}(L)}(3) \leq 2$. In addition, $\{\max(\text{st}(L)) - 2, \max(\text{st}(L)) - 1\} \not\subseteq \text{st}(L)$, otherwise $\{\max(L) - 2, \max(L) - 1, 1\} \subseteq L$ is a component. Now, Table 2 shows the exhaustive list of the possible $\text{st}(L)$ found by computer search given the constraints

(1) $|\text{st}(L)| \leq 13$,
(2) $N_{\text{st}(L)}(3) \leq 2$, and
(3) $\{\max(\text{st}(L)) - 2, \max(\text{st}(L)) - 1\} \not\subseteq \text{st}(L)$

and why each is not a sum graph labeling of P_n. Therefore, $[1, 2] \subseteq S$. ☐

Now, Lemmas 3.17, 3.18, and 3.19 give the following corollary.

Corollary 3.20. *If* $\text{range}(L) = 2n$ *for odd* $n \geq 17$ *and* $a_1 = 1$*, then* $2 \in S$*.*

Now, we show the analog of Corollary 3.20 for $a_1 = 2$.

Lemma 3.21. *If* $\text{range}(L) = 2n$ *for odd* $n \geq 17$ *and* $a_1 = 2$*, then* $4 \in S$*.*

Proof. Suppose otherwise. Because $m \geq 0$, from Corollary 3.5, $k \leq 2$. If $a_2 \geq 6$, then $m = 1$ and $\{2, 3\} \subseteq T \setminus M$. Then, from Corollary 3.5, $\text{range}(L) \geq 2n+1$. Thus, because

$\mathrm{st}(L)$	Why P_n is not a sum graph of L.
$\{1,3,4,5,7,9,13\}$	$\{1,3,5\} \subseteq N(4)$
$\{1,3,4,6,8,9,13\}$	$\lvert N(2n-6)\rvert = \lvert N(2n-4)\rvert = \lvert N(2n-3)\rvert = 1$
$\{1,3,4,6,8,11\}$	$\{1,3,2n-2\}$ is a component.
$\{1,3,5,6,8,10,13\}$	$\{1,5,2n-4\}$ is a component.
$\{1,3,5,7,8,11\}$	$\{1,2n-3\}$ is a component.
$\{1,3,6,7,8,10,12,15\}$	$N(2n-4) = \emptyset$
$\{1,3,6,8,10,11,12,15\}$	$\lvert N(2n-4)\rvert = \lvert N(2n-3)\rvert = \lvert N(2n-2)\rvert = 1$
$\{1,3,5,6,7,10,12,14,17\}$	$\lvert N(2n-6)\rvert = \lvert N(2n-4)\rvert = \lvert N(2n-2)\rvert = 1$
$\{1,3,5,7,10,11,12,15\}$	$\{2n-9,2n-7,2n-4\} \subseteq N(5)$ for $n \geq 9$.
$\{1,3,4,6,8,12,13\}$	$\{1,3,2n\}$ is a component.
$\{1,3,5,6,8,10,14,15\}$	$\{1,3,2n-4\} \subseteq N(5)$
$\{1,3,5,7,8,10,12,16,17\}$	$\lvert N(2n-6)\rvert = \lvert N(2n-4)\rvert = \lvert N(2n)\rvert = 1$
$\{1,3,5,7,8,10,14,15\}$	$\lvert N(2n-6)\rvert = \lvert N(2n-4)\rvert = \lvert N(2n)\rvert = 1$
$\{1,3,5,7,8,12,13\}$	$\lvert N(3)\rvert = \lvert N(2n-4)\rvert = \lvert N(2n)\rvert = 1$
$\{1,3,6,7,10,12,14,15\}$	$N(2n-4) = \emptyset$
$\{1,3,6,7,9,11,14,15\}$	$\{1,6,2n-5,2n\}$ is a component.
$\{1,3,6,8,10,11,14,15\}$	$\{1,2n-4,2n\}$ is a component.
$\{1,3,5,6,9,11,13,16,17\}$	$\{5,6,2n-5\}$ is a cycle.
$\{1,3,5,7,10,11,14,15\}$	$\lvert N(n)\rvert = \lvert N(n-2)\rvert = \lvert N(2n-1)\rvert = 1$ for odd n.

TABLE 2. Remaining possible $\mathrm{st}(L)$ in the proof of Lemma 3.19

$2a_1 = 4 \in T$, it holds that $a_2 \in \{3,5\}$. First, if $a_2 = 3$, then because $a_n \leq \mathrm{range}(L)$, from Lemma 3.8, $n \leq 16$. Thus, $a_2 = 5$.

Because $a_2 = 5$, it follows that $m = 1$, and because $3 \in T \setminus M$, from Corollary 3.5, $k = 1$. From Lemma 3.4, $2n-1 \leq a_n \leq 2n$. Because $3 \in T$, if $2n-1 \in S$, then $N(2n-1) = \emptyset$. Thus, because $\{2n-1, 2\} \subseteq T \setminus M$, it follows that $k \geq 2$. Finally, from Corollary 3.5, $\mathrm{range}(L) \geq 2n+1$. Thus, $2a_1 \in S$. □

Next, we show the analog of Corollary 3.20 for $a_1 = 3$.

Lemma 3.22. *If* $\mathrm{spum}(P_n) = 2n$ *for odd* $n \geq 17$*, and* $a_1 = 3$*, then* $2a_1 \in S$*.*

Proof. Suppose otherwise. Then, as $m \geq 0$, it follows from Corollary 3.5 that $k \leq 2$. Next, from Lemma 3.4, $2n-3 \leq a_n \leq 2n$. But, because $N(a_n) \neq \emptyset$ and $a_{n+1} = a_1 + \mathrm{range}(L) = 2n+3$, it holds that $a_n \neq 2n-3$. Next, because $6 \in T$ and $a_{n+1} = 2n+3$, if $a_n = 2n-2$, then from Lemma 3.4, $\{2n-3\} = T \setminus M$, and $m = 0$. Thus, $[3,5] \subseteq S$. In addition, by setting $t = 6$ (resp. $t = 2n-3, 2n-1$) in Proposition 3.9, $9 \in S$ (resp. $\{2n-4, 2n-6\} \subseteq S$). Because $N(2n-4) \neq \emptyset$, it holds that $7 \in S$. Thus, $\{3,5,2n-6\} \subseteq N(4)$, so $a_n \neq 2n-2$.

Now, if $a_n = 2n-1$ then $4 \in S$, otherwise $N(2n-1) = \emptyset$. In addition, $\{2n-3\} = T \setminus M$, otherwise $N(2n-3) = \emptyset$. Because $N(2n-2) \neq \emptyset$, if $2n-2 \in S$, then $5 \in S$. Therefore, if $\{9, 2n-6\} \cap S \neq \emptyset$, then $\{4,5,2n-6,2n-2,2n-1\}$ is a component. Thus, $\{9, 2n-6\} \subseteq T$, and $\{6, 2n-6, 2n-3\} \subseteq T \setminus M$. Because $k \leq 2$ from Corollary 3.5, we have that $2n-2 \in T$. Then from Lemma 3.4, $\{2n-3, 2n-2\} = T \setminus M$, and $m = 0$. Therefore, $[3,5] \subseteq S$, and $\{6, 2n-5\} \cap T \setminus M = \emptyset$. Thus, by setting $t = 6$ (resp. $t = 2n-2$) in Proposition 3.9, $9 \in S$ (resp. $2n-5 \in S$) which implies $\{5, 2n-5, 2n-1\} \subseteq N(4)$. Therefore, $a_n \neq 2n-1$.

Therefore, $a_n = 2n$. If $a_{n-1} = 2n-1$ then $4 \in S$, as otherwise $N(2n-1) = \emptyset$. Thus, $\mu = 2$, and $\epsilon = 1$. Now, because $n \geq 17$ and from Lemma 3.8, $a_{n-1} \leq 2n-2$. If $a_{n-1} \leq 2n-3$, then from Lemma 3.4, $\{2n-2, 2n-1\} = T \setminus M$. Next, because $m = 0$, it follows that $[3,5] \subseteq S$. Thus, by setting $t = 6$ (resp. $t = 2n-2, 2n-1$) in Proposition 3.9, $9 \in S$ (resp. $\{2n-5, 2n-4\} \subseteq S$). Now, if $7 \in S$ (resp. $8 \in S$), then $\{3, 5, 2n-4\} \subseteq N(4)$ (resp. $\{3, 4, 2n-5\} \subseteq N(5)$). Therefore, $\{7, 8\} \subseteq T$, which implies $\{2n-4, 4, 5, 2n-5\}$ is a component, so $a_{n-1} = 2n-2$. Now, because $N(2n-2) \neq \emptyset$, it holds that $5 \in S$. Next, if $2n-4 \in S$, because $|N(2n-2)| = |N(2n)| = 1$, it follows that $|N(2n-4)| = 2$, which implies $\{4, 7\} \subseteq S$. Now, because $N(4) = \{3, 2n-4\}$, it holds that $9 \in T$, and from Lemma 3.4, $\{6, 2n-1\} = T \setminus M$. Thus, because $9 \notin T \setminus M$, by setting $t = 9$ in Proposition 3.9, $12 \in S$. Now, because $|N(2n-2)| = |N(2n)| = 1$, it holds that $\{3, 4, 5, 7, 2n-4, 2n-2, 2n\}$ is a component. Thus, $2n-4 \in T$, and from Lemma 3.4, $\{2n-4, 2n-1\} = T \setminus M$, and $m = 0$, so $[3,5] \subseteq S$. Thus, because $6, 2n-7 \notin T \setminus M$, by setting $t = 6$ (resp. $t = 2n-4$) in Proposition 3.9, $9 \in S$ (resp. $2n-7 \in S$). Therefore, $\{4, 2n-7, 2n-2\} \subseteq N(5)$. Thus, because $\Delta = 2$, it holds that $2a_1 \in S$. $\square$

Now, we show the analog of Corollary 3.20 for $a_1 \geq 4$.

Lemma 3.23. *If* $\operatorname{spum}(P_n) = 2n$ *and* $a_1 \geq 4$*, then* $2a_1 \in S$*.*

Proof. Suppose otherwise. Then from Corollary 3.5 and Lemma 3.6, $a_1 = 4$, $k = 2$ and $m = 0$. Therefore, $[4,7] \subseteq S$. Because $|[a_n-3, a_n-1] \cup (T \setminus M)| \geq 2$ by the definition of k, it holds that $T \setminus M \subseteq [a_n-3, a_n-1]$. Thus, by setting $t = 8 \notin T \setminus M$ in Proposition 3.9, $12 \in S$. In addition, $a_{n+1} = a_1 + \operatorname{spum}(P_n) = 2n+4$, and from Lemma 3.4, $a_n \in \{2n-1, 2n\}$.

If $a_n = 2n-1$, then $\{7, 2n-1\} = N(5)$. If $2n-3 \in S$, then because $12 \in S$, it follows that $\{2n-1, 5, 7, 2n-3\}$ is a component. If $2n-4 \in S$, then because $8 \in S$, it follows that $N(2n-4) = \emptyset$. Thus, from Corollary 3.5, $T \setminus M = \{2n-4, 2n-3\}$. Now, by setting $t = 2n+2$ in Proposition 3.9, $2n-2 \in S$. If $13 \in S$, then $\{2n, 5, 7, 6, 2n-1\}$ is a component. Therefore, $13 \in T$. Furthermore, because $4 \notin N(5)$, it follows that $9 \in T \setminus M$, so $a_n = 2n$.

Now, by the definition of k, $|[a_n-3, a_n-1] \cap S| = 1$. If $2n-3 \in S$, then $9 \notin S$ as otherwise, $\{2n, 4, 5, 7, 2n-3\}$ is a component. Thus, $9 \in T$, and by setting $t = 9 \notin T \setminus M$ in Proposition 3.9, $13 \in S$. Then, $\{5, 6, 2n-3\} \subseteq N(7)$, so $2n-3 \notin S$. If $2n-2 \in S$, then by setting $t = 2n-1 \notin T \setminus M$ in Proposition 3.9, $a_n - 5 \in S$. Thus, $\{7, 2n-5\} = N(5)$. Because $4 \in S$, but $4 \notin N(5)$, it holds that $9 \in T$. But then because $\{9, 2n-1\} \subseteq T$, it holds that $|N(2n-5)| = |N(2n-2)| = |N(2n)| = 1$, so $2n-2 \notin S$. Therefore, $2n-5 \in S$, and so $|N(5)| = 2$ implies that $a_n - 6 \in T \setminus M$, a contradiction with $T \setminus M \subseteq [a_n-3, a_n-1]$. Thus, $2a_1 \in S$. $\square$

Finally, from Corollary 3.20 and Lemmas 3.21, 3.22, and 3.23, we prove Theorem 1.1.

Proof of Theorem 1.1. The statement is true for $n \leq 15$ from computer search by Li [4]. From Theorem 3.16, $\operatorname{spum}(P_n) \geq 2n$ for odd $n \geq 9$. Now, assume for the sake of contradiction that $\operatorname{range}(L) = 2n$ for odd $n \geq 17$. First, because $m \geq -1$, from Corollary 3.5, $k \leq 3$. Then from Lemma 3.6, $a_1 \leq 5$. Furthermore, from Corollary 3.20 and Lemmas 3.21, 3.22, and 3.23, $2a_1 \in S$. Thus, from Corollary 3.3, $2n = \operatorname{range}(L) \geq 3n - a_1 - 11$, which implies that $n \leq a_1 + 11 \leq 16$. Lastly, from Theorem 3.1, $\operatorname{spum}(P_n) \leq 2n+1$. $\square$

4. The Integral Sum-Diameter of Paths

In this section, we prove Theorem 1.4, from which Theorem 1.2 follows. Currently, the best known lower bound on $\mathrm{isd}(P_n)$ is by Singla, Tiwari, and Tripathi [5], and the best known upper bound is by Li [4].

Theorem 4.1 (Theorem 2.2 of [5] and Proposition 9.6 of [4])**.** *For $n \geq 3$, it holds that*

$$2n-5 \leq \mathrm{isd}(P_n) \leq \begin{cases} 2n-2 & \text{if } n \text{ is even} \\ 2n-3 & \text{if } n \text{ is odd} \end{cases}.$$

For the rest of this section, let $G \cup I_t$ be an integral sum graph of L that satisfies $\mathrm{range}(L) = \mathrm{isd}(G)$. Let $L = \{a_1, a_2, \ldots, a_{n+t}\}$ with $a_1 < \cdots < a_{r-1} < 0 < a_r < \cdots < a_{n+t}$, and let $S \subseteq L$ be the labels of the vertices of G. Additionally, because $-L$ is still an integral sum graph labeling of $G \cup I_t$, we assume without loss of generality that $a_r \leq -a_{r-1}$. Now, define $b_i = a_{i+1} - a_i$ for $1 \leq i \leq n+t-1$, and c_ℓ be the number of $1 \leq i \leq n+t-1$ such that $b_i = \ell$. For each $1 \leq i \leq n+t$, say that a_i and a_{i+1} are consecutive. Finally, denote $N_L(\ell) = N(\ell)$ unless otherwise specified.

Next, we borrow notations from [5], and let

$$\begin{aligned} S_1 &= \{a_1, a_2, \ldots, a_{r-1}\}, \\ S_2 &= \{a_r, a_{r+2}, \ldots, a_{n+t}\}, \\ S_3 &= S_1 + a_r, \text{ and} \\ S_4 &= S_2 - a_r. \end{aligned}$$

In addition, for this section, let $T = [a_1, a_{n+t}] \setminus L$ and $M = S_3 \cup S_4$. Finally, let $\eta = 1$ if $a_{r-1} = -a_r$, and $\eta = 0$ otherwise. Similarly, let $\xi = 1$ if $2a_r \in L$, and $\xi = 0$ otherwise.

4.1. The inequality $\mathrm{isd}(P_n) \geq 2n-4$ for $n \geq 8$.

First, we set a lower bound on $|T \setminus M|$.

Lemma 4.2. *It holds that $|([a_{r-1} - a_r + 1, -a_r - 1] \cup [a_r, 2a_r - 1]) \setminus L| \leq |T \setminus M|$.*

Proof. First, because $S_3 = S_1 + a_r = M \cap \mathbb{Z}_{\leq 0}$,

$$|[a_{r-1} - a_r + 1, -a_r - 1] \setminus S_1| = |[a_{r-1} + 1, -1] \setminus S_3| = |[a_{r-1} + 1, -1] \setminus M|. \tag{4.1}$$

Next, because $S_4 = S_2 - a_r = M \cap \mathbb{Z}_{\geq 0}$,

$$|[a_r, 2a_r - 1] \setminus S_2| = |[0, a_{r-1} - 1] \setminus S_4| = |[0, a_r] \setminus M|. \tag{4.2}$$

Now, from Equations (4.1) and (4.2), it holds that

$$|([a_{r-1} - a_r + 1, -a_r - 1] \cup [a_r, 2a_r - 1]) \setminus L| \leq |[-a_{r-1} + 1, a_r - 1] \setminus M|. \tag{4.3}$$

Now, by the definitions of a_{r-1} and a_r,

$$[a_{r-1} + 1, a_r - 1] \setminus M \subseteq T \setminus M. \tag{4.4}$$

It follows from Equation (4.3) that the lemma holds. $\square$

Now, we generalize Theorem 2.2 of [5].

Proposition 4.3. *It holds that $|S_1 \cap S_3| + |S_2 \cap S_4| = N(a_r) + \xi$.*

Proof. First, because $a_r + a_{r-1} \leq 0$,

$$N(a_r) \cap S_1 = S_1 \cap S_3. \tag{4.5}$$

Similarly, it holds that

$$N(a_r) \cap S_2 = (S_2 \cap S_4) \setminus \{a_r\}. \tag{4.6}$$

Finally, as $a_r \in S_2 \cap S_4$ if and only if $2a_r \in S_2$, it holds that

$$|(S_2 \cap S_4) \setminus \{a_r\}| = |S_2 \cap S_4| - \xi. \tag{4.7}$$

Now, because $S_1 \sqcup S_2 = L$, from Equations (4.5), (4.6), and (4.7),

$$|N(a_r)| = |S_1 \cap S_3| + |S_2 \cap S_4| - \xi. \tag{4.8}$$

□

Next, we strengthen Proposition 4.3.

Proposition 4.4. *It holds that* $\sum_{i=1}^{4} |S_i| - |\bigcup_{i=1}^4 S_i| = |N(a_r)| + \xi + \eta$.

Proof. First, $S_3 \cap S_4 = \{0\}$ if and only if $a_{r-1} = -a_r$. Thus, it holds that

$$|S_3 \cap S_4| = \eta. \tag{4.9}$$

Next, because $a_{r-1} + a_r \leq 0$, it follows from the definitions of S_1, S_2, S_3, S_4 that

$$S_1 \cap S_4 = \emptyset,\ S_2 \cap S_3 = \emptyset,\ S_1 \cap S_2 = \emptyset. \tag{4.10}$$

Thus, from Proposition 4.3 and Equations (4.9) and (4.10),

$$\sum_{i=1}^{4} |S_i| - |\bigcup_{i=1}^{4} S_i| = |S_1 \cap S_3| + |S_2 \cap S_4| + |S_3 \cap S_4| = |N(a_r)| + \xi + \eta. \tag{4.11}$$

□

Now, we use Proposition 4.4 to refine Theorem 2.2 of [5] to $\operatorname{isd}(P_n)$.

Lemma 4.5. *It holds that* $\operatorname{isd}(P_n) = 2(n+t) + |T \setminus M| - |N(a_r)| - \xi - \eta - 1$.

Proof. First, because $\bigcup_{i=1}^4 S_i = (S_1 \sqcup S_2) \cup (S_3 \cup S_4) = L \cup M$, it holds that

$$[a_1, a_{n+t}] \setminus \bigcup_{i=1}^{4} S_i = [a_1, a_{n+t}] \setminus (L \cup M) = T \setminus M. \tag{4.12}$$

Now, because $\bigcup_{i=1}^4 S_i \subseteq [a_1, a_{n+t}]$, from Equation (4.12),

$$|[a_1, a_{n+t}]| - |\bigcup_{i=1}^{4} S_i| = |T \setminus M|. \tag{4.13}$$

Therefore, from Equation (4.13) and Proposition 4.4, it holds that

$$\begin{aligned}
\operatorname{isd}(P_n) = a_{n+t} - a_1 = |[a_1, a_{n+t}]| - 1 &= |\bigcup_{i=1}^{4} S_i| + |T \setminus M| - 1 \\
&= \sum_{i=1}^{4} |S_i| + |T \setminus M| - |N(a_r)| - \xi - \eta - 1 \qquad (4.14) \\
&= 2(n+t) + |T \setminus M| - |N(a_r)| - \xi - \eta - 1.
\end{aligned}$$

□

If $\operatorname{isd}(P_n) = 2n - 5$, then Lemma 4.5 results in the following corollary.

Corollary 4.6. *If* $\mathrm{isd}(P_n) = 2n-5$*, then* $t = 0$*,* $T \subseteq M$*,* $|N(a_r)| = 2$*, and* $\xi = \eta = 1$*.*

Next, we show that if $[a, b]$ and $[c, d]$ are disjoint intervals that share sufficiently many elements with L, then another interval is disjoint from L. Let $\tau = |([a, b] \sqcup [c, d]) \setminus L|$.

Lemma 4.7. *If* $[a, b]$ *and* $[c, d]$ *are disjoint intervals such that* $\min(b-a, d-c) \geq 2+\tau$*, then*

$$([2+\tau+a-d, -2-\tau+b-c] \cup [2+\tau+c-b, d-a-2-\tau]) \cap L = \emptyset.$$

Proof. We first show that $[2+\tau+a-d, -2-\tau+b-c] \cap L = \emptyset$. Suppose otherwise, and let $\ell = 2+\tau+a-d+k \in [2+\tau+a-d, -2-\tau+b-c] \cap L$. Because $\ell \leq -2-\tau+b-c$,

$$0 \leq k \leq d-c+b-a-4-2\tau. \tag{4.15}$$

Consider $([c, d]+\ell) \cap [a, b] = [c+\ell, d+\ell] \cap [a, b]$. From Equation (4.15),

$$c+\ell = 2+\tau+a+c-d+k \leq b-2-\tau, \text{ and} \tag{4.16}$$

$$d+\ell = 2+\tau+a+k \geq a+2+\tau. \tag{4.17}$$

Thus, from Equations 4.16 and 4.17, either $[b-2-\tau, b] \subseteq ([c, d]+\ell)$, or $[a, a+2+\tau] \subseteq [c, d]+\ell$, or $[c, d]+\ell \subseteq [a, b]$. Then

$$|[a, b] \cap ([c, d]+\ell)| \geq 3+\tau. \tag{4.18}$$

Now, because $|([a, b] \cup [c, d]) \setminus L| \leq \tau$, and if $\{t, t+\ell\} \subseteq L$ then $t \in N(\ell)$ from Equation (4.18), it holds that $N(\ell) \geq 3$, a contradiction as $G = P_n$. Lastly, swapping $[a, b]$ and $[c, d]$ gives $[2+\tau+a-d, -2-\tau+b-c] \cap L = \emptyset$. ☐

We now show that if $T \setminus M$, and $\{-2a_r, -a_r, a_r, 2a_r\} \subseteq L$, then $a_r = 1$.

Lemma 4.8. *If* $T \subseteq M, \xi = \eta = 1$*,* $t \leq 1$*,* $-2a_r \in L$*, and* $n \geq 9$*, then* $a_r = 1$*.*

Proof. Because P_n is a integral sum graph of $-L$, assume without loss of generality that $a_{n+t} \geq -a_1$. Now, because $T \subseteq M$, $-2a_r \in L$, and from Lemma 4.2,

$$[-2a_r, -a_r] \cup [a_r, 2a_r] \subseteq L. \tag{4.19}$$

Next, because $t \leq 1$, we have $\{2a_r-1, -2a_r+1\} \cap S \geq 1$. Thus, because $N(2a_r-1) \neq \emptyset$ or $N(-2a_r+1) \neq \emptyset$, and $a_{n+t} \geq -a_1$, from Equation (4.19), $a_n \geq 3a_r - 1$.

Now, because $a_n \geq 3a_r - 1 \geq 4a_r - 2$ for $a_r \geq 3$, by applying Lemma 4.7 on intervals $[-2a_r, -a_r]$ and $[a_r, 2a_r]$, it holds that

$$[2a_r+2, 4a_r-2] \subseteq T. \tag{4.20}$$

Thus, if $a_r \geq 4$, then $\{2a_r+2, 3a_r+2\} \subseteq T$, so $2a_r \in T \setminus M$. Thus, $a_r \leq 3$.

Now, if $a_r = 3$, because $a_n \geq 3a_r - 1 = 8$, and setting $a_r = 3$ in Equation (4.20) implies $[8, 10] \subseteq T$, we have $a_n \geq 11$. Thus, because $\{7, 8\} \cap T \setminus M = \emptyset$, and $[8, 10] \subseteq T$, we have $\{7, 11\} \subseteq L$. But,by Equation (4.19), $\{-3, -4, 4\} \subseteq N(7)$. Thus, $a_r \neq 3$.

Finally, if $a_r = 2$, by applying Lemma 4.7 on intervals $[-4, -2]$ and $[2, 4]$, it follows that $\{-6, 6\} \cap L = \emptyset$. Now, because $n \geq 9$, and $a_n \geq -a_1$, it follows that $a_n \geq 7$. Thus, $6 \in T \subseteq M$, which implies $8 \in L$. Now, if $7 \in L$ then $N(-4) = \{2, 7, 8\}$. Thus, $7 \in T$. However, if $5 \in L$, then $\{-3, -2, 3\} \subseteq N(5)$. Thus, $5 \in T \setminus M$. Therefore, $a_r = 1$. ☐

We now show that if $\mathrm{isd}(P_n) = 2n-5$ and $n \geq 11$, then $a_r = 1$.

Corollary 4.9. *If* $\mathrm{isd}(P_n) = 2n-5$ *for* $n \geq 9$*, then* $a_r = 1$*.*

Proof. Because P_n is a integral sum graph of $-L$, by applying Corollary 4.6 to L and $-L$, we have $\xi = \eta = 1$, $T \subseteq M$, $t = 0$, and $-2a_r \in L$. Thus, by Lemma 4.8, $a_r = 1$. $\square$

Now, we bound $\operatorname{isd}(P_n)$ from below assuming $a_r = 1$, $a_{r-1} + a_r = 0$, and $2a_r \in L$.

Lemma 4.10. *If $T \subseteq M$ and $a_r = \xi = \eta = 1$, then $\operatorname{range}(L) \geq 3n - 12$.*

Proof. Because $a_r = \xi = 1$, it follows that $\{1, 2\} \subseteq L$. Now, if $i \neq r$, and $b_i = 1$, then $a_i \in N(1)$. Thus, $c_1 \leq 3$ because $|N(1)| \leq 2$.

Likewise, if $b_i = 2$ and $i \neq r + 1$, then $a_i \in N(2)$. Thus, $c_2 \leq 3$, because $|N(2)| \leq 2$. Therefore,

$$\operatorname{range}(L) = \sum_{i=1}^{n+t-1} b_i \geq c_1 \cdot 1 + c_2 \cdot 2 + (n + t - 1 - c_1 - c_2) \cdot 3 \geq 3n - 12. \tag{4.21}$$ $\square$

We now use Lemmas 4.9 and 4.10 to show that $\operatorname{isd}(P_n) \geq 2n - 4$ when $n \geq 8$.

Lemma 4.11. *If $n \geq 8$, then $\operatorname{isd}(P_n) \geq 2n - 4$.*

Proof. Suppose otherwise. Then $\operatorname{range}(L) = 2n - 5$ by Theorem 4.1. First, $\operatorname{isd}(P_n) \geq 2n - 4$ by exhaustive search for $n = 8$. Next, if $n \geq 9$, then $a_r = \xi = \eta = 1$ from Corollary 4.6 and Corollary 4.9. Thus, $\operatorname{range}(L) \geq 3n - 12 > 2n - 5$ from Lemma 4.10. $\square$

4.2. **The inequality** $\operatorname{isd}(P_n) \geq 2n - 3$ **for** $n \geq 8$**.** Now, we state a corollary of Lemma 4.5 if $\operatorname{isd}(P_n) = 2n - 4$.

Corollary 4.12. *If $\operatorname{isd}(P_n) = 2n - 4$ and $n \geq 9$, then $t = 0$, and $|T \setminus M| = \xi + \eta - 1$.*

Proof. From Lemma 4.5, if $\operatorname{isd}(P_n) = 2n-4$, then $2t + |T \setminus M| \geq |N(a_r)| + \xi + \eta - 3$. Because $\max(\xi, \eta) \leq 1$, and $|N(a_r)| \leq 2$, it holds that $t = 0$. Now, if $|N(a_r)| \leq 1$, then $T \subseteq M$ and $\xi = \eta = 1$. Thus, from Lemma 4.8 and Lemma 4.10, $a_r = 1$ and $\operatorname{range}(L) \geq 3n - 12 > 2n - 4$, because $n \geq 9$. Thus, $|N(a_r)| = 2$, and $|T \setminus M| = \xi + \eta - 1$ from Lemma 4.5. $\square$

Now, let $k = |([-2a_r, -a_r] \cup [a_r, 2a_r]) \cap T|$. In addition, let

$$J_1 = ([a_1, -2a_r - 1] \cap L) + 2a_r, \text{ and}$$
$$J_2 = ([2a_r + 1, a_{n+t}] \cap L) - 2a_r.$$

We now prove the analog of of Corollary 3.3 for $\operatorname{isd}(P_n)$.

Lemma 4.13. *If $-2a_r \in L$ or $2a_r \in L$, then $\operatorname{range}(L) \geq 3n + 3t + k - 2a_r - 11$.*

Proof. First, $J_1 \cap J_2 = \emptyset$. Additionally, because $S \sqcup T = [a_1, a_n]$,

$$\begin{aligned} |J_1 \cup J_2| &= |([a_1, -2a_r - 1] \cup [2a_r + 1, a_{n+t}]) \cap L| \\ &= n + t - |([-2a_r, -a_r] \cup [a_r, 2a_r]) \cap L| \\ &= n + t - 2(a_r + 1) + k. \end{aligned} \tag{4.22}$$

Now, because $(S_3 \cap S_1) - a_r \subseteq N(a_r)$, and $S_4 \cap S_2 \subseteq N(a_r) \cup \{a_r\}$, but $(S_1 - a_r) \cap S_2 = \emptyset$,

$$|(S_3 \cap S_1)| + |S_4 \cap S_2| \leq |N(a_r) \cup \{a_r\}| \leq 3. \tag{4.23}$$

First, if $2a_r \in L$, then $(J_1 \cap S_1) - 2a_r \subseteq N(2a_r) \setminus \{-a_r\}$ and $J_2 \cap S_2 \subseteq N(2a_r) \cup \{2a_r\}$. Next, because $(S_1 - 2a_r) \cap S_2 \neq \emptyset$ and $-a_r \in N(2a_r)$ if and only if $\eta = 1$,

$$|(J_1 \cap S_1)| + |J_2 \cap S_2| \leq |N(a_r) \cup \{2a_r\} \setminus \{-a_r\}| \leq 3 - \eta. \tag{4.24}$$

Equivalently, if $-2a_r \in L$, then $J_1 \cap S_1 \subseteq N(-2a_r) \cup \{-2a_r\}$ and $J_2 \cap S_2 \subseteq N(-2a_r) \setminus \{a_r\}$, but $a_r \in N(-2a_r)$ if and only if $\eta = 1$. Thus, Equation (4.24) holds.

Now, because $S_1 \sqcup S_2 \sqcup T = [a_1, a_{n+t}]$, and $S_3 \cap S_4 \subseteq \{0\}$, from Equation (4.23),

$$\begin{aligned} |T \cap (S_3 \cup S_4)| &= |S_3 \cup S_4| - |S \cap (S_3 \cup S_4)| \\ &\geq (n + t - \eta) - |(S_3 \cap S_1)| - |(S_4 \cap S_2)| \\ &\geq (n + t - \eta) - 3. \end{aligned} \tag{4.25}$$

Likewise, because $S_1 \sqcup S_2 \sqcup T = [a_1, a_{n+t}]$ and $J_1 \cap J_2 = \emptyset$, from Equations (4.22) and (4.24),

$$\begin{aligned} |T \cap (J_1 \cup J_2)| &= |J_1 \cup J_2| - |S \cap (J_1 \cup J_2)| \\ &= n + t - 2(a_r + 1) + k - |S_1 \cap J_1| - |S_2 \cap J_2| \\ &\geq n + t - 2a_r + k - 5 + \eta. \end{aligned} \tag{4.26}$$

Additionally, because $(J_1 \cap S_3) \sqcup (J_2 \cap S_4) \subseteq N(a_r)$, it follows that

$$|J_1 \cap S_3| + |J_2 \cap S_4| \leq |N(a_r)| \leq 2. \tag{4.27}$$

Lastly, because $(J_1 \cap S_3) \cap (J_2 \cap S_4) = J_1 \cap J_2 = \emptyset$, from Equations (4.25), (4.26), and (4.27),

$$\begin{aligned} |T| &\geq |T \cap (S_3 \cup S_4 \cup J_1 \cup J_2)| \\ &= |T \cap (S_3 \cup S_4)| + |T \cap (J_1 \cup J_2)| - |T \cap (S_3 \cup S_4) \cap (J_1 \cup J_2)| \\ &\geq |T \cap (S_3 \cup S_4)| + |T \cap (J_1 \cup J_2)| - |(S_3 \cup S_4) \cap (J_1 \cup J_2)| \\ &\geq (n + t - \eta - 3) + (n + t - 2a_r + k - 5 + \eta) - (|J_1 \cap S_3| + |J_2 \cap S_4|) \\ &\geq 2n + 2t + k - 8 - 2a_r - 2 = 2n + 2t + k - 2a_r - 10. \end{aligned} \tag{4.28}$$

Because $|T| = \text{range}(L) + 1 - n - t$, from Equation (4.28), $\text{range}(L) \geq 3n + 3t + k - 2a_r - 11$. $\square$

Now, we strengthen Corollary 4.12.

Lemma 4.14. *If* $\text{isd}(P_n) = 2n - 4$ *and* $n \geq 13$, *then* $T \subseteq M$.

Proof. Suppose otherwise. Because P_n is a integral sum graph of $-L$, by applying Corollary 4.12 on L and $-L$, we have $|T \setminus M| = \xi = \eta = 1$, $t = 0$, and $-2a_r \in L$. Additionally, assume without loss of generality that $a_n \geq |a_1|$. First, from Lemma 4.2,

$$|([-2a_r, -a_r] \cup [a_r, 2a_r]) \setminus L| = k \leq 1. \tag{4.29}$$

Thus, $|\{2a_r - 1, -2a_r + 1\} \cap S| \geq 1$. Without loss of generality, let $2a_r - 1 \in L$. Therefore, because $N(2a_r - 1) \neq \emptyset$, and $a_n \geq |a_1|$, we have $a_n \geq 3a_r - 1$.

Now, suppose that $k = 0$. Because $a_n \geq 3a_r - 1 \geq 2a_r + 2$ for $a_r \geq 3$, by setting $[a, b] = [-2a_r, -a_r]$ and $[c, d] = [a_r, 2a_r]$ in Lemma 4.7,

$$[2a_r + 2, 4a_r - 2] \subseteq T. \tag{4.30}$$

If $a_r \geq 5$, then from Equation (4.30), $\{2a_r + 2, 2a_r + 3\} \subseteq T \setminus M$, so $a_r \leq 4$.

If $a_r = 4$, from Equation (4.30), $[10, 14] \subseteq T$. Thus, $T \setminus M = \{10\}$ because $|T \setminus M| = 1$. Now, $\{9, 15\} \subseteq L$ because $\{9, 11\} \cap T \setminus M = \emptyset$. Then $\{-8, -7, -6\} \subseteq N(15)$. Thus, $a_r \leq 3$.

If $a_r = 3$, from Equation (4.30), $[8, 10] \subseteq T$. First, if $7 \in L$, then $N(7) = \{-3, -4\}$ because $\Delta = 2$. Thus, $[8, 13] \subseteq T$ because $[4, 6] \cap N(7) = \emptyset$ and $[8, 10] \subseteq T$. Thus, $\{8, 9\} \subseteq T \setminus M$. So, $7 \in T$. Now, because $|T \setminus M| = 1$ and $10 \in T$, it follows that $T \setminus M = \{7\}$. Thus,

$[11,13] \subseteq L$ because $[8,10] \cap T \setminus M = \emptyset$. As a result, $\{3,11,12\} \subseteq N(-6)$. Thus, $a_r \leq 2$. Now, because $\xi = 1$ and $a_r \leq 2$, from Lemma 4.13, $\text{range}(L) = 2n-4 \geq 3n-16$, so $n \leq 12$.

Next, suppose that $k = 1$. First, because $|T \setminus M| = 1$, we have $T \setminus M \subseteq [-a_r, a_r]$. Now, if $n \geq 4$, then because $a_n \geq 3a_r - 1 \geq 2a_r + 3$, by Lemma 4.7 on $[-2a_r, -a_r]$ and $[a_r, 2a_r]$ with $\tau = 1$, we have $[2a_r+3, 4a_r-3] \subseteq T$. Now, because $\{a_r+3, 2a_r+3, 3a_r-3, 4a_r-3\} \cap T \setminus M = \emptyset$, it must be that $\{a_r+3, 3a_r+3, 3a_r-3, 5a_r-3\} \subseteq L$, and $\{a_r, 3a_r+3, 5a_r-3\} \subseteq N(-2a_r)$. Thus, $n \leq 3$, and it follows that $n \leq 12$ by Lemma 4.13, because $k = \xi = 1$. $\square$

We now further strengthen Corollary 4.12 with the following result:

Lemma 4.15. *If* $\text{isd}(P_n) = 2n-4$ *for* $n \geq 13$*, then* $\eta = 1$ *and* $\xi = 0$.

Proof. Suppose otherwise. Then, from Corollary 4.12 and Lemma 4.14, $T \subseteq M$, $\xi = 1$, and $t = \eta = 0$. Then $a_{r-1} < -a_r \leq -1$. Thus, $-1 \in T \subseteq M$, which implies $a_{r-1} = -(a_r+1)$. Therefore, as $T \subseteq M$, from Lemma 4.2,

$$[-2a_r, -a_r-1] \cup [a_r, 2a_r] \subseteq L. \tag{4.31}$$

Because $t = 0$, it follows that $N(2a_r - 1) \neq \emptyset$. Therefore, $a_1 \leq -3a_r$ or $a_n \geq 3a_r - 1$ or both. First if $a_n \geq 3a_r - 1$, then from Equation (4.31) and Lemma 4.7 with intervals $[-2a_r, -a_r]$, and $[a_r, 2a_r]$ and $\tau = 0$, it holds that $[2a_r+3, 4a_r-2] \cap L = \emptyset$.

Now, if $a_r \geq 4$, then $a_n \geq 3a_r - 1 \geq 2a_r + 3$, so $[2a_r+3, 4a_r-2] \subseteq T$. Then, because $T \subseteq M$, we have $\{a_r+3, 3a_r+3, 3a_r-3, 5a_r-3, 3a_r-2, 5a_r-2\} \subseteq L$. Thus, $\{a_r+3, 3a_r-3, 3a_r-2\} \subseteq N(2a_r)$. Therefore, $a_r \leq 3$, so by Lemma 4.13, $n \leq 12$ because $k = 1$ and $a_{r-1} = -(a_r+1)$. As a result, $a_1 \leq -3a_r$.

Now, if $a_r \geq 4$ then $a_1 \leq -3a_r \leq -2a_r + 3$, so from Equation (4.31) and Lemma 4.7 with intervals $[-2a_r, -a_r] \cup [a_r, 2a_r]$ and $\tau = 0$, we have $[-4a_r+2, -2a_r-3] \subseteq T$. Then, because $T \subseteq M$, then $\{a_r+3, 3a_r-3, 3a_r-2\} \subseteq N(2a_r)$. Therefore, $a_r \leq 3$, so by Lemma 4.13, $n \leq 12$, because $k = 1$ and $a_{r-1} = -(a_r+1)$. $\square$

Lemmas 4.2, 4.14, and 4.15, result in the following corollary.

Corollary 4.16. *If* $\xi = t = 0$*,* $\eta = 1$*, and* $T \subseteq M$*, then* $[-2a_r+1, a_r] \cup [a_r, 2a_r-1] \subseteq L$.

Now, we show that if $\text{isd}(P_n) = 2n-4$, then $a_r \leq 3$.

Lemma 4.17. *If* $\xi = t = 0$*,* $\eta = 1$*, and* $T \subseteq M$ *for* $n \geq 13$*, then* $a_r \leq 3$.

Proof. Suppose otherwise. First, because P_n is a sum graph of $-L$, assume that $a_{n+t} \geq |a_1|$. From Corollary 4.16, $2a_r - 1 \in L$. Then, because $t = 0$, it holds that $N(2a_r - 1) \neq \emptyset$. Thus, since $a_n \geq |a_1|$, we have $a_n \geq 3a_r - 1$. As a result, because $a_r \geq 4$, we have $a_n \geq 3a_r - 1 \geq 2a_r + 2$, and by setting intervals $[-2a_r+1, a_r] \cup [a_r, 2a_r-1]$ with $\tau = 0$ in Lemma 4.7, $3a_r \in [2a_r+2, 4a_r-4] \subseteq T$, which implies $2a_r \in T \setminus M$ because $\xi = 0$. Thus, $a_r \leq 3$. $\square$

Next, we show that if $\text{isd}(P_n) = 2n-4$, then $a_r \leq 2$.

Lemma 4.18. *If* $\xi = t = 0$*,* $\eta = 1$*,* $T \subseteq M$*, and* $-2a_r \notin L$ *for* $n \geq 13$*, then* $a_r \leq 2$.

Proof. Suppose otherwise. Thus, from Corollary 4.16 and Lemma 4.18, $a_r = 3$ and $[-5,-3] \cup [3,5] \subseteq L$. Because $t = 0$ from Corollary 4.12, $N(5) \neq \emptyset$. Then, because P_n is a integral sum graph labeling of $-L$, assume $a_n \geq 8$. Thus, because $6 \in T \subseteq M$ from Lemma 4.15, $9 \in L$. Now, by setting $\tau = 0$ with intervals $[-5,-3] \cup [3,5]$ in Lemma 4.7, $8 \in T \subseteq M$, so $11 \in L$.

Next, because $9 \in L$, it follows that $\{-4,-5\} \subseteq N(9)$. Now, $\{-4,-5,9\}$ forms a cycle if $-9 \in L$. Thus, $-9 \notin L$ but $-2a_r = 6 \notin L$. Thus, $a_1 = -5$ because $T \subseteq M$. In addition, because $\Delta = 2$, we have $\{-4,-5\} = N(9)$, which implies $[12,14] \cap L = \emptyset$, because $[3,5] \cap N(9) = \emptyset$. If $a_n \geq 15$, then $[12,14] \subseteq T \subseteq M$, so $\{10,15,16\} \subseteq L$. As a result, $\{4,10,11\} \subseteq N(5)$, so $a_n < 15$. Now, $a_n = 11$ because $[12,14] \cap L = \emptyset$. Thus, $a_r \leq 2$. □

Now, we show that if $\mathrm{isd}(P_n) = 2n-4$, then $a_r = 1$.

Lemma 4.19. *If $\xi = t = 0$, $\eta = 1$, and $T \subseteq M$ for $n \geq 13$, then $a_r \leq 1$.*

Proof. Suppose otherwise. By Lemma 4.18, $a_r = 2$. Then, from Corollary 4.16, $T \subseteq M$. For $1 \leq i \leq n-2$, if $b_i, b_{i+1} \neq 1$, then $\{a_{i+1}-1, a_{i+1}+1\} \cap T \setminus M \neq \emptyset$. Thus, if $b_i = 2$, then $b_{i+1} = 1$ or $b_{i-1} = 1$ or both, $\{a_i, a_{i-1}\} \cap N(3) \neq \emptyset$. However, because $3 \in L$, then $c_2 + c_3 \leq 3$.

Next, $\{a_i+1, a_i+3\} \cap T \setminus M \neq \emptyset$ if $b_i = 4$ and $i \neq r-1$. Therefore, because $b_{r-1} = 4$, we have $c_4 = 1$. It follows that $\mathrm{range}(L) \leq 4c_4 + 3c_3 + 2c_2 + (n-1-c_2-c_3-c_4) \leq n+8$. As a result, $\mathrm{isd}(P_n) = 2n-4 \leq n+8$, which implies $n \leq 12$ or $a_r = 1$ or both. □

Now, we show that $\mathrm{isd}(P_n) \geq 2n-3$ for $n \geq 8$.

Lemma 4.20. *If $n \geq 8$, then $\mathrm{isd}(P_n) \geq 2n-3$.*

Proof. Suppose otherwise. From Lemma 4.11, $\mathrm{isd}(P_n) = 2n-4$. By exhaustive computer search, $\mathrm{isd}(P_n) \geq 2n-3$ if $8 \leq n \leq 12$. Thus, $n \geq 13$. Because P_n is a sum graph labeling of $-L$, assume that $a_n \geq |a_1|$. In addition, from Lemmas 4.12 and 4.15 on L and $-L$, we have $\xi = t = 0$, $\eta = 1$, $T \subseteq M$, $|N(a_r)| = 1$, and $-2a_r \notin L$. Thus, from Lemmas 4.17, 4.18, and 4.19, $a_r = 1 = |a_{r-1}|$ for $n \geq 13$. Because $n \geq 13$ and $a_n \geq |a_1|$, it follows that $a_n \geq 11$.

Now, if $b_i \geq 3$, then $\{a_i+1, a_i+2\} \cap T \setminus M \neq \emptyset$. Therefore, $b_i \leq 2$ for $1 \leq i \leq n-1$. Furthermore, $c_1 = 2$, because $|N(1)| = 2$. Thus, let $b_j = b_k = 1$ with $j < k$.

Now, because $\xi = 0$, it holds that $2 \in T \subseteq M$. Then, because $a_n \geq 11$, it holds that $3 \in L$. If $k \neq n-1$ and $k-j = 1$, then $\{-3,-1,1\} \in N(a_j+1)$. However, if $k \neq n-1$ and $k-j \neq 1$, then $\{-1, a_j+1, -3, a_k+1\}$ form a component. Therefore, $k = n-1$.

Likewise, if $-3 \in L$, then $j = 1$. Thus, because $a_n \geq 11$ and $k = n-1$, it follows that $\{1,3,5\} \subseteq L$. Because $j = 1$ and $k = n-1$, it holds that $\{1,3,5\} \subseteq N(a_1)$. Therefore, $-3 \notin L$, so $a_1 = -1$. As $\mathrm{range}(L) = 2n-4$ and $k = n-1$, we have $\{2n-5, 2n-6\} \subseteq L$.

First, suppose that $2n-7 \in L$. Then $N(2n-5) = \{-1\}$, and $N(2n-7) = \{1\}$ because $\min(L) = a_1 = -1$ and $2 \notin L$. Thus, because $\Delta = 2$, and $\{-1,1\} \subseteq N(2n-6)$, it follows that $\{-1, 1, 2n-7, 2n-6, 2n-5\}$ forms a component. Therefore, $2n-7 \notin L$. As a result, $2n-8 \in L$, otherwise $2n-8 \in T \setminus M$. Because $\min(L) = -1$ and $2n-7 \notin L$, we have $|N(2n-5)| = |N(2n-6)| = 1$. Now, $|N(2n-8)| = 2$, because $G = P_n$. Therefore, $j = n-3$. Then, $\{-1, 1, 3, 2n-9, 2n-8, 2n-6, 2n-5\}$ forms a component. Thus, $\mathrm{isd}(P_n) \geq 2n-3$. □

Now, Lemma 4.20 is sufficient to proof Theorem 1.2

Proof of Theorem 1.2. From Proposition 7.2 of Li [4], it holds that $\mathrm{isd}(P_n) \leq \mathrm{ispum}(P_n)$. Thus, the lower and upper bounds of Theorem 1.2 are shown by Lemma 4.20 and Theorem 7.3 of Singla, Tiwari, and Tripathi [5], respectively. □

4.3. **The inequality $\mathrm{isd}(P_n) \geq 2n+1$ for $n \geq 27$.** Now, we show that for odd n, $\mathrm{isd}(P_n) \neq 2n-3$. First, we state the following corollary of Lemma 4.5.

Corollary 4.21. *If $\mathrm{isd}(P_n) = 2n-3$, then $2t + |T \setminus M| + 2 = \xi + \eta + |N(a_r)|$.*

Next, we show that if $\text{isd}(P_n) = 2n - 3$, then L has no isolated vertices.

Lemma 4.22. *If* $\text{isd}(P_n) = 2n - 3$ *for* $n \geq 10$*, then* $t = 0$*.*

Proof. Suppose otherwise. Then, as P_n is a integral sum graph of $-L$, by applying Corollary 4.21 to L and $-L$, we have $T \subseteq M$, $|N(a_r)| = 2$, $\xi = \eta = t = 1$, and $-2a_r \in L$. Thus, from Lemmas 4.8 and 4.10, $a_r = 1$, and $2n - 3 = \text{isd}(P_n) \geq 3n - 12$, so $n \leq 9$. □

We now further strengthen Corollary 4.21 by showing that $\xi = 0$.

Lemma 4.23. *If* $\text{isd}(P_n) = 2n - 3$ *for* $n \geq 27$*, then* $\xi = 0$*.*

Proof. Suppose otherwise. First, if $a_r \leq 9$, from Lemma 4.13, $n \leq 26$. Thus, $a_r \geq 10$, and from Corollary 4.21, $|T \setminus M| \leq 2$. Then, from Lemma 4.2,

$$|[a_{r-1} - a_r + 1, -a_r - 1] \cup [a_r, 2a_r] \setminus L| \leq 2. \tag{4.32}$$

From Equation (4.32), there exists $\ell \in \{2a_r - 1, -2a_r + 1, 2a_r - 2\}$ in L. Then, because $t = 0$, and $N(\ell) \neq \emptyset$, we have $\max(|a_1|, a_n) \geq 3a_r - 2$.

First suppose that equality holds in Equation (4.32). From Lemma 4.7, with $\tau = 2$, and intervals $[a_{r-1} - a_r + 1, -a_r - 1] \cup [a_r, 2a_r]$, and because $\max(|a_1|, a_n) \geq 3a_r - 2 \geq 2a_r + 5$, we have $[-3a_r + a_{r-1} + 5, -2a_r - 5] \subseteq T$ or $[2a_r + 5, 3a_r - a_{r-1} - 5] \subseteq T$ or both. Then $2a_r + 5 \in T \setminus M$ or $-2a_r - 5 \in T \setminus M$ or both. But, $|T \setminus M \cap [a_{r-1}, a_r]| = 2$ and $|T \setminus M| \leq 2$. Thus, $|[a_{r-1} - a_r + 1, -a_r - 1] \cup [a_r, 2a_r] \setminus L| \leq 1$.

Next, suppose $|[a_{r-1} - a_r + 1, -a_r - 1] \cup [a_r, 2a_r] \setminus L| = 1$. From Lemma 4.7 with $\tau = 1$, and intervals $[a_{r-1} - a_r + 1, -a_r - 1] \cup [a_r, 2a_r]$, and because $\max(|a_1|, a_n) \geq 3a_r - 2 \geq 2a_r + 4$, it must be that $[-3a_r + a_{r-1} + 4, -2a_r - 4] \subseteq T$ or $[2a_r + 4, 3a_r - a_{r-1} - 4] \subseteq T$. As a result, $|\{-2a_r - 4, -2a_r - 5, 2a_r + 4, 2a_r + 5\} \cap T \setminus M| \geq 2$. However, $|T \setminus M \cap [a_{r-1}, a_r]| = 1$ and $|T \setminus M| \leq 2$. Thus, $[a_{r-1} - a_r + 1, -a_r - 1] \cup [a_r, 2a_r] \setminus L = \emptyset$.

Then, from Lemma 4.7, with $\tau = 0$ and intervals $[a_{r-1} - a_r + 1, -a_r - 1] \cup [a_r, 2a_r]$, we have $([-3a_r + a_{r-1} + 3, -2a_r - 3,] \cup [2a_r + 3, 3a_r - a_{r-1} - 3]) \cap L = \emptyset$. As a result, and also because $\max(|a_1|, a_n) \geq 3a_r - 2 \geq 2a_r + 4$, it must be that $[2a_r + 3, 2a_r + 5] \subseteq T \setminus M$ or $[-2a_r - 3, 2a_r - 5] \subseteq T \setminus M$ or both. Therefore, $\xi = 0$. □

We now show if $\text{isd}(P_n) = 2n - 3$ then $|N(a_r)| = 2$.

Lemma 4.24. *If* $\text{isd}(P_n) = 2n - 3$ *for* $n \geq 27$*, then* $|N(a_r)| = 2$*.*

Proof. Suppose otherwise. From Corollary 4.21, Lemmas 4.22 and 4.23, $t = 0$, $|N(a_r)| = 1$, $T \subseteq M$ and $\eta = 1$. Now, because P_n is a sum graph labeling of $-L$, we have $-2a_r \in L$ from Lemma 4.23 on $-L$. As a result, if $n \geq 13$, then $a_r = 1$ from Lemmas 4.17, 4.18, and 4.19.

Now, by applying Corollary 4.21 on $-L$, it holds that $|N(a_{r-1})| = 1$. Thus, because $G = P_n$ and $t = 0$, if $a_i \notin \{1, -1\}$ then $|N(a_i)| = 2$. In addition, because $|N(a_r)| = 1$ and $T \subseteq M$, we have $c_1 = 1$, and $c_2 = n - 2$. Thus, one of a_1, a_n is even. Without loss of generality, let a_n be even. Since $1 \in L$, all even labels are in S_2, and all labels in S_1 are odd.

However, as $a_n = \max(L)$, it follows that $N(a_n) \subseteq S_1$. Let $N(a_n) = \{a_j, a_k\}$ with $j < k$. Then, because $\{a_j, a_k + a_n\} \subseteq L$, all odd labels between a_j and $a_k + a_n$ must be in L. Thus, because $a_j \leq a_k$, it follows that $\{a_j, a_j + 2, a_k + 2\} \subseteq N(a_n - 2)$. As a result, $a_n - 2 \notin L$ and a_n is the only even label. Then, $|N(a_i)| \leq 1$ for for $1 \leq i \leq n - 2$, so $|N(a_r)| = 2$. □

We now show that if $\text{isd}(P_n) = 2n - 3$ then $|T \setminus M| = \eta = 1$.

Lemma 4.25. *If* $\text{isd}(P_n) = 2n - 3$ *for* $n \geq 27$*, then* $|T \setminus M| = \eta = 1$*.*

Proof. Suppose otherwise. From Corollary 4.21 and Lemmas 4.22, 4.23, and 4.24, $|T \setminus M| = \eta = 0$. Thus, $a_{r-1} = -(a_r+1)$ because $|a_{r-1}| \neq a_r$ and $-1 \in T \subseteq M$. Now, from Lemma 4.2,

$$([-2a_r, -1-a_r] \cup [a_r, 2a_r-1]) \subseteq L. \tag{4.33}$$

Next, from Lemma 4.7 with $\tau = 0$ and intervals $[-2a_r, -1-a_r] \cup [a_r, 2a_r-1]$,

$$([-4a_r+3, -2a_r-3] \cup [2a_r+3, 4a_r-3]) \cap L = \emptyset. \tag{4.34}$$

First, because $N(-2a_r) \neq \emptyset$ from Lemma 4.22 and Equation (4.33), $\max(|a_1|, a_n) \geq 3a_r$. Now, if $a_r \geq 6$, because $\max(|a_1|, a_n) \geq 3a_r \geq 2a_r+3$, it must be that $[-4a_r+3, -2a_r-3] \subseteq T$ or $[2a_r+3, 4a_r-3] \subseteq T$ or both. Thus, $|\{-2a_r-3, 2a_r+3\} \cap T \setminus M| \geq 1$. Therefore, $a_r \leq 5$. Finally, because $k \geq 1$ from $-a_r \notin L$, and $-2a_r \in L$ from Equation (4.33), it follows from Lemma 4.13 that $2n-3 = \mathrm{isd}(P_n) \geq 3n - 2a_r - 10$, so $n \leq 17$. $\square$

Now, we use Lemmas 4.22, 4.24, 4.23, and 4.25 to bound the value of a_r.

Lemma 4.26. *If* $\mathrm{isd}(P_n) = 2n-3$ *for* $n \geq 27$*, then* $a_r \leq 4$.

Proof. Suppose otherwise. From Lemma 4.25, $|T \setminus M| = \eta = 1$. Thus, from Lemma 4.2,

$$|([-2a_r+1, -a_r] \cup [a_r, 2a_r-1]) \setminus L| \leq 1. \tag{4.35}$$

As $t = 0$ by Lemma 4.22, and $2a_r - 1 \in S$ or $-2a_r+1 \in S$ or both, and because P_n is a sum graph labeling of $-L$, assume without loss of generality that $\max(|a_1|, a_n) = a_n \geq 3a_r - 1$. In addition, because $3a_r - 1 > 2a_r + 3$, from Equation (4.35) and by setting $\tau = 1$, with intervals $[-2a_r+1, -a_r] \cup [a_r, 2a_r-1]$ in Lemma 4.7, $[2a_r+3, 4a_r-5] \subseteq T$. Thus, $3a_r \in T$. However, from Lemmas 4.23 and 4.25, $2a_r = T \setminus M$. Therefore, because $[a_{r-1}, a_r] \cap T \setminus M = \emptyset$,

$$[-2a_r+1, -a_r] \cup [a_r, 2a_r-1] \subseteq L, \tag{4.36}$$

so because $a_n \geq 3a_r - 1 \geq 2a_r + 2$, by setting $\tau = 0$ and disjoint intervals $[-2a_r+1, -a_r] \cup [a_r, 2a_r-1]$ in Lemma 4.7, $[2a_r+2, 4a_r-4] \subseteq T$. For $a_r \geq 6$, it holds that $\{2a_r, 2a_r+2\} \subseteq T \setminus M$. Thus, $a_r = 5$, so $[2a_r+2, 4a_r-4] = [12, 16] \in T$. Then because $\{11, 12\} \cap T \setminus M = \emptyset$, we have $11, 17 \in L$, and thus, $\{-5, -6, 6\} \subseteq N(11)$. Therefore, $a_r \leq 4$. $\square$

We now strengthen Lemma 4.26 by showing that $a_r \neq 4$.

Lemma 4.27. *If* $\mathrm{isd}(P_n) = 2n-3$ *for* $n \geq 27$*, then* $a_r \leq 3$.

Proof. Suppose otherwise. Then by Lemmas 4.25 and 4.26, $a_r = 4$, and $|T \setminus M| = \eta = 1$. Thus, from Lemma 4.2, $|([-7, -4] \cup [4, 7]) \setminus L| \leq 1$. From Lemma 4.22 and because $|\{7, -7\} \cap L| \geq 1$, it follows that $N(7) \neq \emptyset$ or $N(-7) \neq \emptyset$ or both. Therefore, because P_n is a sum graph labeling of $-L$, we have $\max(|a_1|, a_n) = a_n \geq 11$.

First, suppose that $[-7, -4] \cup [4, 7] \subseteq L$. Because $a_n \geq 11$, by setting $\tau = 0$ and disjoint intervals $[-7, -4]$ and $[4, 7]$ in Lemma 4.7, $[10, 12] \subseteq T$. Then, because $8 \in T$ from Lemma 4.23, and $|T \setminus M| = 1$, it holds that $T \setminus M = \{8\}$. Thus, as $\{10, 11, 12\} \cap T \setminus M = \emptyset$, we have $\{10, 11, 12\} \subseteq M$, so $[14, 16] \subseteq L$. Now, $9 \notin L$, otherwise $\{5, 6, 7\} \subseteq N(9)$. Thus, because $9 \notin T \setminus M$, we have $13 \in L$. Because $|N(13)| \leq 2$ and $\{-7, -6\} \subseteq N(13)$, we must have $-8 \notin N(13)$, and thus $-8 \notin L$. However, if $-8 \in T$, as $-8 \notin T \setminus M$, we have that $-12 \in L$, which implies that $\{5, 6, 7\} \subseteq N(-12)$. Therefore, $-8 \notin T$ so $a_1 = -7$, and as $n \geq 27$, we have that $a_n \geq 20$. As $[4, 7] \cap N(13) = \emptyset$ and $a_n \geq 20$, it follows that $[17, 20] \subseteq T$. Because $17 \notin T \setminus M$, it holds that $21 \in L$. However, then $\{-7, -6, -5\} \subseteq N(21)$.

Thus, $[-7,-4] \cup [4,7] \not\subseteq L$. Now, from Lemma 4.7 with $\tau = 1$ and disjoint intervals $[-7,-4]$ and $[4,7]$, and because $a_n \geq 11$, it holds that $11 \in T$, so $15 \in L$. Furthermore, from Lemma 4.23, $8 \in T$, and thus, $12 \in L$. We now consider each of $([-7,4] \cup [4,7]) \setminus L$.

First, because $\eta = 1$, we have $\{-4,4\} \subseteq L$. Now, $6 \notin T$, as otherwise $10 \in L$, and $\{-5,-6,5\} \subseteq N(10)$. Similarly, $-6 \notin T$, as otherwise $-10 \in L$, so $\{5,6,15\} \subseteq N(-10)$. Next, suppose that $5 \in T$. Then $9 \in L$, so $\{9,12\} \subseteq N(-5)$. As $N(-5) \leq 2$, we have $\{14,15\} \cap N(5) = \emptyset$, so $\{10,14\} \subseteq T$, and thus, $10 \in T \setminus M$. Thus, $5 \notin T$ Now, suppose that $-5 \in T$. Then, $-9 \in L$, which implies $-8 \in T$, because $-8 \notin L$ by applying Lemma 4.23 on $-L$. Thus, $\{-12,-9,7\} \subseteq N(5)$. Therefore, $-5 \notin T$.

Additionally, if $7 \in T$, because $11 \in T$, we have $7 \in T \setminus M$, a contradiction. Note that $-7 \notin T$ as otherwise $-11 \in L$ and $\{5,6,7\} \subseteq N(-11)$. Therefore, if $-7 \notin L$, then $a_1 = -6$. However, as $\{-4,-5,6\} \subseteq N(9)$ and $\{-4,-5,-6\} \subseteq N(10)$, it follows that $\{9,10\} \subseteq T$, so $\{13,14\} \subseteq L$. Then, because $|N(7)| \leq 2$, and $\{5,6\} \subseteq N(7)$, it follows that $[12,15] \cap N(7) = \emptyset$, so $[19,22] \cap L = \emptyset$. Now, as $15 \in L$ and $[9,11] \subseteq T$, we have $\{-4,-5,-6\} \cap N(15) = \emptyset$. As $N(15) \neq \emptyset$, it holds that $\min(N(15)) \geq 4$, so $a_n \geq 19$. However, because $[19,22] \cap L = \emptyset$, it holds that $[19,22] \subseteq T$, and thus, $[16,18] \subseteq L$, which implies $\{7,12,13\} \subseteq N(5)$. Thus, $a_r \leq 3$. ☐

Lemma 4.28. *If* $\operatorname{isd}(P_n) = 2n-3$ *for* $n \geq 27$*, then* $a_r \leq 2$*.*

Proof. Suppose otherwise. Then, $a_r = 3$ by Lemma 4.27. Now, from Lemma 4.25, $|T \setminus M| = \eta = 1$. Thus, from Lemma 4.2, $|([-5,-3] \cup [3,5]) \setminus L| \leq 1$.

Then, because P_n is a sum graph labeling of $-L$, assume without loss of generality that $[3,5] \subseteq L$. First, if $b_i = 5$, then $|\{a_i+1, a_i+4\} \cap T \setminus M| \geq 1$. Because $|T \setminus M| = 1$, it holds that $c_5 \leq 1$. Similarly, if $b_i \geq 6$ but $i \neq r-1$, then $a_i+1, a_i+4 \in T$ and $a_i+2, a_i+5 \in T$. Thus, because $|T \setminus M| = 1$ and $b_{r-1} = 6$, we have $c_6 = 1$. Next, because $[3,5] \subseteq L$, it follows that $4, 8$ are not consecutive. In addition, from Lemma 4.23, $6 \notin L$. Thus, because $|N(4)| \leq 2$ and $|N(3)| \leq 2$, we have $c_4 \leq 2$, and $c_3 \leq 2$.

Finally, we bound c_2. First, if $b_1 = 2$ and $b_{n-1} = 2$, then $\{a_1+1, a_n-1\} \subseteq T \setminus M$, a contradiction. Thus, assume that $b_{n-1} \neq 2$. Now, if $b_i = 2$ and $b_{i+1} = 1$, then $a_i \in N(3)$. Next, if $b_i = 2$ and $b_{i+1} = 2$, then $a_i \in N(4)$. In addition, if $b_i = 2$ and $b_{i+1} \geq 3$, then $\{a_i+1, a_i+4\} \cap T \setminus M \neq \emptyset$. Now, because $|N(3)|, |N(4)| \leq 2$, and $|T \setminus M| = 1$,

$$c_2 \leq (2-c_3) + (2-c_4) + (1-c_5) = 5 - c_3 - c_4 - c_5. \tag{4.37}$$

Thus, $\max(c_3, c_4) \leq 2$ and $c_5 \leq 1$, so from Equation (4.37), it holds that

$$\begin{aligned} \operatorname{range}(L) = \sum_{\ell=1}^{\max(b_i)} \ell \cdot c_\ell &= 6 + 5c_5 + 4c_4 + 3c_3 + 2c_2 + (n - 1 - \sum_{\ell=2}^{\max(b_i)} c_\ell) \\ &= n + 4 + 4c_5 + 3c_4 + 2c_3 + c_2 \\ &\leq n + 4 + 4c_5 + 3c_4 + 2c_3 + (5 - c_3 - c_4 - c_5) \leq n + 18. \end{aligned} \tag{4.38}$$

Therefore, because $2n-3 = \operatorname{range}(L) \leq n+18$, it holds that $n \leq 21$. ☐

We now show that $a_r = 1$ if $\operatorname{isd}(P_n) = 2n-3$ for $n \geq 27$.

Lemma 4.29. *If* $\operatorname{isd}(P_n) = 2n-3$ *for* $n \geq 27$*, then* $a_r = 1$*.*

Proof. Suppose otherwise. Then, by Lemma 4.28, $a_r = 2$. First, if $b_i = 4$, with $i \neq r-1$, then $\{a_i+1, a_i+3\} \subseteq T$, which results in an element of $T \setminus M$. Thus, because $|T \setminus M| \leq 1$, and $b_{r-1} = 4$, we have $c_4 \le 2$. In addition, $|\{-3, 3\} \cap L| \ge 1$, as otherwise $\{-1, 1\} \subseteq T \setminus M$, a contradiction to $|T \setminus M| = 1$. Thus, because $N(\pm 3) \le 2$, we have $c_3 \le 3$. Now, if $b_1 = b_{n-1} = 2$, then $\{a_1 + 1, a_n - 1\} \subseteq T \setminus M$, a contradiction. Thus, assume that $b_{n-1} \ne 2$. Then, if $b_i \ne 1$ and $b_{i+1} \ne 1$, then $\{a_{i+1} - 1, a_{i+1} + 1\} \subseteq T$, which results in an element of $T \setminus M$. Therefore, for each $b_i = 2$, either $b_{i+1} = 1$ or $a_i + 1 \in T \setminus M$.

Now, because $|T \setminus M| = 1$, and $|N(3)| \le 2$, it holds that

$$c_2 \le (3 - c_3) + (1 - c_4) = 4 - c_3 - c_4. \tag{4.39}$$

Therefore, because $c_3 \le 3$ and $c_4 \le 2$, it holds that

$$\begin{aligned} \text{range}(L) &\le 4c_4 + 3c_3 + 2c_2 + (n - 1 - c_4 - c_3 - c_2) \\ &\le n - 1 + 3c_4 + 2c_3 + (4 - c_3 - c_4) \\ &= n + 3 + 2c_4 + c_3 \le n + 10. \end{aligned} \tag{4.40}$$

Thus, $\text{isd}(P_n) = 2n - 3 \le n + 10$, from which $n \le 13$, a contradiction. ☐

We now show our final result for odd n.

Lemma 4.30. *For odd $n \ge 27$, it holds that* $\text{isd}(P_n) \ge 2n - 2$.

Proof. Suppose otherwise. Then, by Lemmas 4.25 and 4.29, $|a_{r-1}| = a_r = 1$. In addition, from Lemmas 4.23 and 4.24, because $|N(1)| = 2$ and $2 \notin L$, we have $c_1 = 2$. Furthermore, from Lemma 4.25, $|T \setminus M| = 1$, so $c_3 = 1$, and there is no i such that $b_i \ge 4$. Therefore, let $b_f = 1$, $b_g = 1$, and $b_h = 3$, where $f < g$. In addition, for every $i \notin \{f, g, h\}$, we have $b_i = 2$.

In addition, for $n \ge 14$, there are at least 3 indices i for which $b_i = b_{i+1} = b_{i+2} = 2$. Thus, $\{4, -4, 6, -6\} \cap L = \emptyset$, as otherwise $|N(\pm 6)|, |N(\pm 4)| \ge 2$. It follows that either $-3 \in L$ or $\min(L) = -1$, and either $3 \in L$ or $\max(L) = 1$. In addition, because $6 \notin L$ and $b_h = 3$, there is at most one index i such that $\{b_i, b_{i+1}\} = \{1, 2\}$, otherwise $|N(3)| > 2$ or $|N(-3)| > 2$.

First, we consider $h \notin [f, g]$. Since $-L$ is also a sum graph labeling of P_n, without loss of generality let $h > g$. If $g \ne f + 1$, then $\{b_f, b_{f+1}\} = \{b_{g-1}, b_g\} = \{1, 2\}$. Therefore, $g = f + 1$. Now suppose that $f \ne 1$. Then, because $h > g$, it holds that $b_{f-1} = 2$, which implies $-3 \notin L$, as otherwise $\{-1, 1, -3\} \subseteq N(a_g)$. Thus, it follows that $\min(L) = a_1 = -1$. Then because $t = 0$ from Lemma 4.22, it holds that $b_{n-1} = 1$, because otherwise a_n would be isolated, for $\min(L) = -1$. However, $h > g$, which implies $b_{n-1} \ne 1$. Thus, $f = 1$. Now, if $3 \in L$, then $h = g + 1$, as otherwise $\{-1, 1, 3\} \subseteq N(a_g)$. However, this forces $|N(3)| = |N(-3)| = |N(a_n)| = 1$, a contradiction to $G = P_n$. Thus, $3 \notin L$ and thus, $1 = \max(L) = a_n$, i.e., 1 is the only positive label of L. It follows from $f = g - 1 = 1$ and $2 \notin L$ that $a_3 = a_1 + 2$ and a_1 have exactly one neighbor. Therefore, because $|N(-1)| = |N(1)| = |N(a_2)| = 2$, it follows that $\{-1, 1, a_1, a_2, a_3\}$ is a component.

Therefore, $h \in [f, g]$. First, if $f \ne 1$ and $g \ne n - 1$, we have $\{b_{f-1}, b_f\} = \{1, 2\} = \{b_g, b_{g+1}\}$, a contradiction, because there is at most one i such that $\{b_i, b_{i+1}\} = \{1, 2\}$. Therefore, either $f = 1$ or $g = n - 1$. Assume without loss of generality that $g = n - 1$. First, suppose that $f = 1$. Then either $h = g - 1$ or $h = f + 1$, as otherwise, $\{b_{f+1}, b_f\} = \{1, 2\} = \{b_{g-1}, b_g\}$. Assume without loss of generality that $h = g - 1$. Now, because $\{-6, -4, 4, 6\} \cap L = \emptyset$, we have $|a_1|, a_n \ge 8$. Thus, $\{1, 3, 5, a_1 + 1, a_1 + 3, a_1 + 5\} \subseteq L$, so $\{1, 3, 5\} \subseteq N(a_1)$. Therefore, $f \ne 1$, which forces $f + 1 = h = g - 1$, as otherwise $\{b_{f-1}, b_f\} = \{1, 2\}$ and either $\{b_f, b_{f+1}\} = \{1, 2\}$ or $\{b_{g-1}, b_g\} = \{1, 2\}$. Because $b_{n-4} = 2$, we have $-3 \notin L$, as otherwise $\{-3, -1, 3\} \subseteq N(a_{n-2})$. Therefore, $a_1 = -1$, and $\{f, g, h\} = \{n - 3, n - 2, n - 1\}$, which corresponds to the set of all odds from $[-1, 2n-9]$, combined with $\{2n-8, 2n-5, 2n-4\}$. For odd n, $\{n-4, n-2, 2n-4\}$ each have exactly one neighbor, which contradicts $G = P_n$. $\square$

We now proceed to the proof of Theorem 1.4.

Proof of Theorem 1.4. We verify the case when $n \leq 26$ through computer search. Now, the lower bounds of the theorem are a result of Lemmas 4.20 and 4.30. The upper bounds are a result of Theorem 4.1. $\square$

5. The Sum-Diameter of Paths

We prove Theorem 1.3. The current best bound on $\mathrm{sd}(P_n)$ is by Li.

Theorem 5.1 (Proposition 9.4 of [4])**.** *For $n \geq 3$, it holds that*

$$2n - 3 \leq \mathrm{sd}(P_n) \leq 2n - 2.$$

For the rest of this section, let L be a sum graph labeling of a graph $G \cup I_m$ that satisfies $\mathrm{range}(L) = \mathrm{sd}(G)$. Let $S = \{a_1, a_2, \ldots, a_n\} \subseteq \mathbb{N}$ be the labels of the vertices of G such that $a_1 < a_2 < \cdots < a_n$. We first cite Li [4] on the labels that must be in S if $\mathrm{range}(L) = 2n - 3$.

Lemma 5.2 (Remark 7.8 of [4])**.** *If* $\mathrm{range}(L) = 2n - 3$*, then* $[a_1, 2a_1] \subseteq S$*.*

In addition, following the proof of Lemma 3.2 directly after defining T, M, and N analogously as Section 3 gives us the following lower bound on $\mathrm{sd}(G)$.

Corollary 5.3. *If* $[a_1, 2a_1] \subseteq S$*, then* $\mathrm{sd}(G) \geq 3n - a_1 - 4 - 4\Delta + \delta$*.*

Next, we give a lower bound on n with respect to a_1 when $\mathrm{range}(L) = 2n-3$ and $n - a_1 \geq 4$.

Lemma 5.4. *If* $\mathrm{range}(L) = 2n - 3$ *and* $n - a_1 \geq 4$*, then* $n \geq 2a_1$*.*

Proof. By Lemma 5.2, it holds that $[a_1, 2a_1] \subseteq S$. Now, because $\mathrm{range}(L) = 2n - 3$, it holds that $\max(S) \leq 2n - 3$, otherwise $\max(S)$ would be isolated. Therefore,

$$|S \cap [2a_1 + 1, 2n - 3]| = n - (a_1 + 1) = n - a_1 - 1. \tag{5.1}$$

As a result,

$$\begin{aligned} |S \cap [2a_1 + 1, n + a_1 + 1]| &= |S \cap [2a_1 + 1, 2n - 3 - (n - a_1 - 4)]| \\ &\geq |S \cap [2a_1 + 1, 2n - 3]| - (n - a_1 - 4) = 3. \end{aligned} \tag{5.2}$$

Thus, if $a_1 + 2a_1 \geq n + a_1 + 1$, then $N(a_1) \geq 3$. As a result, $n \geq 2a_1$. $\square$

Next, we give an upper and lower bound on a_1 when $\mathrm{range}(L) = 2n - 3$.

Lemma 5.5. *If* $\mathrm{range}(L) = 2n - 3$*, then* $n - 2 \geq a_1 \geq n - 8$*.*

Proof. By Corollary 5.3, $\mathrm{sd}(P_n) = 2n - 3 \geq 3n - a_1 - 11$. Therefore, $a_1 \geq n - 8$. In addition, if $a_1 \geq n - 1$, then $a_n \geq a_1 + (n - 1) = 2n - 2$. Thus, $\mathrm{range}(L) \geq 2n - 2 + a_1 - a_1 \geq 2n - 2$. As a result, $n - 2 \geq a_1$. $\square$

We now proceed to the proof of Theorem 1.3.

Proof of Theorem 1.3. From Theorem 5.1, it suffices to show that $\mathrm{sd}(P_n) \neq 2n-3$ when $n \geq 7$. Thus, suppose for the sake of contradiction that $\mathrm{sd}(P_n) = 2n-3$ and $n \geq 7$. Then by Lemma 5.5, it holds that $n-2 \geq a_1 \geq n-8$.

Now, if $n-4 \geq a_1 \geq n-8$, then it follows from Lemma 5.4 that $n \geq 2a_1 \geq 2n-16$. Thus, $n \leq 16$. From exhaustive computer search, we check that the statement holds when $n \leq 16$.

Now, if $a_1 = n-3$, then $\max(L) = 3n-6$. Because all $s \in S$ must satisfy $n-3+s \leq 3n-6$, it must hold that $s \leq 2n-3$. In addition, because $|S| = n$, it must hold that

$$|[n-3, 2n-3] \setminus S| = 1. \tag{5.3}$$

Now, if $2n-3 \in S$, then $\{2n-3, n\} \subseteq N(n-3)$. Therefore, $\{n-1, n-2\} \cap N(n-3) = \emptyset$, and thus, $2n-4, 2n-5 \notin S$. But if so, then $|[n-3, 2n-3] \setminus S| \geq 2$. Thus $2n-3 \notin S$. However, from Equation (5.3), it holds that $S = [n-3, 2n-4]$. Then, $N(n-2) = \{n-3, 2n-4\}$, and thus $[n-1, 2n-3] \notin N(n-2)$. Therefore $[2n-3, 3n-7] \notin L$. But then, since $|N(2n-5)| = |N(2n-4)| = 1$, we have $\{2n-4, n-2, n-3, n-1, 2n-5\}$ is a component, a contradiction as sought.

Finally, if $a_1 = n-2$, then because $\max(L) = 3n-5$, any $s \in S$ should satisfy $s \leq 3n-5-(n-2) = 2n-3$. Because $|S| = n$, it must be that $S = [n-2, 2n-3]$. Then $N(n-2) = \{n-1, 2n-3\}$, and thus $[n, 2n-4] \notin N(n-2)$, which implies $[2n-2, 3n-6] \notin L$. However, then $\{2n-3, n-2, n-1, 2n-4\}$ is a component, a contradiction as sought. $\square$

Acknowledgements

This research was conducted as a part of the PRIMES-USA program, hosted by the Department of Mathematics at MIT. The authors thank Tanya Khovanova, Slava Gerovitch, and Pavel Etingof for their advice. Last but not least, they thank Felix Gotti for reviewing and providing feedback on this research.

References

[1] J. Goodell et al. Sum graphs. *unpublished* (1990).
[2] F. Harary. Sum graphs and difference graphs. *Congr. Numer.* 72 (1990), pp. 101–108.
[3] F. Harary. Sum graphs over all integers. *Discrete Math.* 124 (1994), pp. 99–105.
[4] R. Li. The Spum and Sum-Diameter of Graphs: Labelings of Sum Graphs. *Discrete Math.* 345.5 (2022), p. 112806.
[5] S. Singla, A. Tiwari, and A. Tripathi. Some results on the spum and the integral spum of graphs. *Discrete Math.* 344 (2021), P3.32.

A. Bora, William P. Clements High School, Sugar Land, TX, 77479
Email address: `bora.aryan@gmail.com`

Y. Choi, Harvard University, Cambridge, MA, 02138
Email address: `ychoi@college.harvard.edu`

L. Tang, Interlake High School, Bellevue, WA, 98008
Email address: `lucastang06@gmail.com`