The PRIMES 2022 problem set
General math problems

G1. Consider a plane passing through the midpoints of two opposite edges of a regular tetrahedron. The projection of the tetrahedron to this plane is a quadrilateral of area A with one of the angles 60°. Find the surface area of the tetrahedron.

Solution. Let A, B, C and D be the vertices of the tetrahedron and assume the plane P described in the problem intersects $\overline{AB}$ and $\overline{CD}$ at their midpoints M and N. Let A_1, B_1, C_1 and D_1 be the projections of A, B, C and D to P.

First, we observe that $A_1B_1C_1D_1$ is an isosceles trapezoid. In fact, since $\overline{NA} = \overline{NB}$ and $\angle ANA_1 = \angle BNB_1$, we know that $\overline{NA_1} = \overline{NB_1}$. In particular, this implies $\overline{NM} \perp \overline{A_1B_1}$ since M is also the midpoint of $\overline{A_1B_1}$. Similarly, we can see that $\overline{MN} \perp \overline{C_1D_1}$. As a result, we can conclude that $\overline{A_1D_1} = \overline{C_1B_1}$.

Next, we denote $\alpha := \angle AMA_1$ and $\ell := \overline{AB}$. The area of $A_1B_1C_1D_1$ is

$$A = \frac{\overline{MN}}{2}(\overline{A_1B_1} + \overline{C_1D_1}) = \frac{\ell}{2\sqrt{2}}\left(\overline{AB}\cos\alpha + \overline{CD}\cos\left(\frac{\pi}{2} - \alpha\right)\right)$$
$$= \frac{\ell^2}{2\sqrt{2}}(\cos\alpha + \sin\alpha),$$

so the surface area of the tetrahedron is

$$\sqrt{3} \cdot \ell^2 = \frac{2\sqrt{6}}{\cos\alpha + \sin\alpha}A.$$

Finally, we find out the value of $\cos\alpha + \sin\alpha$. Since $\angle AMA_1 = \alpha$, we know $\angle DND_1 = \frac{\pi}{2} - \alpha$. Then, we have

$$\overline{A_1B_1} - \overline{C_1D_1} = \ell \cdot \cos\alpha - \ell \cdot \sin\alpha = \ell \cdot (\cos\alpha - \sin\alpha).$$

On the other hand, using the fact that one of the angles of $A_1B_1C_1D_1$ is $60^\circ = \frac{\pi}{3}$, WLOG say $\angle D_1A_1B_1 = \frac{\pi}{3}$, we can derive

$$\overline{A_1B_1} - \overline{C_1D_1} = \overline{A_1D_1} = \frac{2}{\sqrt{3}}\overline{MN} = \frac{2\ell}{\sqrt{6}}.$$

Therefore, we get

$$\cos\alpha - \sin\alpha = \frac{2}{\sqrt{6}},$$

which implies $2\cos\alpha \cdot \sin\alpha = \frac{1}{3}$ and hence $\cos\alpha + \sin\alpha = \frac{2}{\sqrt{3}}$. Consequently, we have that the surface area of the tetrahedron is

$$\sqrt{3} \cdot \ell^2 = \frac{2\sqrt{6}}{\cos\alpha + \sin\alpha}A = 3\sqrt{2}A.$$

G2. For an m-digit number A and $(n-m)$-digit number B let $A \circ B$ be the n-digit number obtained by concatenation of A and B (where we allow the leftmost digit to be zero). For example, if $m = 2, n = 5$, $A = 23$, $B = 045$, then $A \circ B = 23045$ and $B \circ A = 04523$.

From now on assume that $m = 2$. Let k be a 2-digit number, and consider the equation

$$\frac{B \circ A}{A \circ B} = k$$

with $A > 0$ and any $n \geq 3$. It is clear that if $X := A \circ B$ is a solution of this equation then so is $X \circ X$, $X \circ X \circ X$, etc. We say that a solution X is *primitive* if it is not obtained in this way, by concatenating a smaller solution with itself several times.

(a) Find all primitive solutions for $k = 9$ and $k = 15$.

(b) Describe all primitive solutions for general k. Are there finitely many?

Solution. Consider the decimal

$$x = 0.ABAB.... = \frac{A \circ B}{10^n - 1}.$$

Then $100x = A.BAB....$ So we get

$$100x - A = kx,$$

thus

$$x = \frac{A}{100 - k}.$$

So we get

$$\frac{A}{100 - k} = \frac{A \circ B}{10^n - 1}.$$

In particular,

$$\frac{A}{100 - k} < \frac{A + 1}{100},$$

i.e.,

$$k(A + 1) < 100.$$

Thus all solutions are obtained by running through pairs of 2-digit positive numbers k, A such that $k(A + 1) < 100$ and $d := \frac{100-k}{GCD(100-k,A)}$ is coprime to 10. For each such k, A, let r be the multiplicative order of 10 modulo d (i.e., in $(\mathbb{Z}/d\mathbb{Z})^\times$). Then $n = qr$, $q \geq 1$, so

$$A \circ B = \frac{A(10^{qr} - 1)}{100 - k}.$$

Primitive solutions correspond to $q = 1$, so there are finitely many. There is a unique primitive solution for each A, k, which is defined by the equality $\frac{A}{100-k} = 0.\overline{X}$.

In particular, for $k = 9$ we have $1 \leq A \leq 10$, so we have the primitive solutions $A \cdot Y$ where $Y = 010989$. For $k = 15$ we get $A = 5$ and only one primitive solution $X = 0588235294117647$.

G3. Let m be a fixed positive integer, and consider the following game. At each move, you pick uniformly at random an integer $0 \leq k \leq m$. Then you score k points, but only if k does not exceed the smallest previously picked number (otherwise you don't score any points on that move). For example, if $m = 3$ and your random numbers are $2, 3, 1, 2, 1, 0, 3, ...$ then you score only on the 1st, 3rd and 5th move and don't score anything after the 5th move, so you total score is $2 + 1 + 1 = 4$.

(i) How much will you score on average if you play indefinitely?

(ii) Let $a(n, m)$ be the average amount you score in n steps. Find a closed formula for $a(n, 1)$ and $a(n, 2)$.

(iii) Find a closed formula for $a(n, m)$.

Solution. Let $a(n) := a(n, m)$ and $a_r(n)$ be the average amount you score in n moves if you choose the number s at random from $0, ..., m$ but score s points only if $s \leq r$. Then we have $a_m(n) = a(n)$, and

$$(m+1)a_r(n) = (m-r)a_r(n-1) + \sum_{k=0}^{r}(k + a_k(n-1)) =$$

$$(m-r+1)a_r(n-1) + \frac{r(r+1)}{2} + \sum_{k=0}^{r-1} a_k(n-1).$$

with $a_0(n) = 0$, $a_r(0) = 0$. So for $b_r(n) = r - a_r(n)$ we get the homogeneous equation

$$(m+1)b_r(n) = (m-r+1)b_r(n-1) + \sum_{k=0}^{r-1} b_k(n-1)$$

with initial condition $b_r(0) = r$. From this it is easy to deduce the following formula:

$$b_r(n) = \sum_{s=1}^{r} c_{rs} \left(\frac{m+1-s}{m+1}\right)^n$$

for some numbers c_{rs} determined recursively. For example, $b_1(n) = (\frac{m}{m+1})^n$, so $a_1(n) = 1 - (\frac{m}{m+1})^n$. In fact, it is easy to show by induction that $c_{rs} = 1$ for all r, s, so

$$b_r(n) = \sum_{s=1}^{r} \left(\frac{m+1-s}{m+1}\right)^n,$$

thus

$$a_r(n) = r - \sum_{s=1}^{r} \left(\frac{m+1-s}{m+1} \right)^n .$$

So the answer is

$$a(n,m) = m - \sum_{k=1}^{m} \left(\frac{k}{m+1} \right)^n$$

In particular, $a_m(\infty) = m$ (the average score if you play indefinitely).

G4. A street is lit by n street lights arranged in a row. If one of them burns out but its neighbors are still working[1], the Department of Public Works (DPW) does not do anything. However, once two consecutive lights are out of order, the DPW immediately replaces the light bulbs in all broken lights. For example:

○ ○ ○ ○ ○

○ ○ ● ○ ○

do nothing

○ ○ ● ○ ●

still do nothing

○ ● ● ○ ●

replace all

(i) What is the chance that the DPW will have to replace k lights, if lights break independently and with equal probability?

(ii) What is the average number of lights that they have to replace in each repair?

Compute the answers for $n = 9$ and $k = 4$ with two digits precision after the decimal point.

Solution. The chance that the arrangement does not require repair after m lights burn out is

$$r_m = \frac{\binom{n-m+1}{m}}{\binom{n}{m}}.$$

So the chance they have to replace k lights is

$$p_k = r_{k-1} - r_k.$$

The average number of lights they have to replace is

$$N = \sum k(r_{k-1} - r_k) = r_0 + r_1 + r_2 + ...$$

[1]Both neighbors, or only one if it is the first or the last light.

For $n = 9, k = 4$ we get

$$p_4 = \frac{25}{84} \approx 0.297619, \;\; N = \frac{93}{28} \approx 3.32.$$

G5. (i) Describe an algorithm to find the closed ball (disk) of smallest radius containing a given finite set of points (x_i, y_i), $i = 1, ..., n$, in $\mathbb{R}^2$.

(ii) Do the same for points (x_1, y_i, z_i), $i = 1, ..., n$, in $\mathbb{R}^3$.

(iii) Show that the ball in (i),(ii) is unique.

Solution. (iii) The ball is unique because the intersection of two distinct balls of the same radius is contained in a ball of smaller radius.

(i) Run through pairs of points (P, Q) and check if the circle with diameter PQ contains all other points inside. If so, we are done. Otherwise run through triples of points forming an acute-angled triangle and check if the circumscribed circle contains all other points. For one of them it must be so, and this is then the answer.

(ii) Same as (ii) but there is an extra step - need to run through quadruples of points whose convex hull contains the center of the circumscribed sphere.

Advanced math problems

M1. Suppose one picks uniformly at random an n by n matrix (a_{ij}) of zeros and ones with odd determinant. What is the probability p that $a_{11} = 0$?

(i) Compute the answer for $n = 2, 3$.

(ii) Compute the answer for general n.

Solution. We need to count invertible matrices over $\mathbb{F}_2$ with $a_{11} = 0$. The total number of invertible matrices is $N = 2^{n(n-1)/2} \prod_{i=1}^{n}(2^i - 1)$, so it suffices to compute the number of matrices in which $a_{11} = 1$. Thus

$$A = \begin{pmatrix} 1 & v \\ w & B \end{pmatrix},$$

where v is a row vector, w a column vector, B a matrix (all of size $n - 1$). It is easy to see that

$$\det A = \det(B - w \otimes v).$$

Thus we can choose w, v arbitrarily, then choose $B' = B - w \otimes v$ to be invertible. The number of such possibilities is

$$N_1 = 2^{2(n-1)+\frac{(n-1)(n-2)}{2}} \prod_{i=1}^{n-1}(2^i - 1).$$

So the number of invertible matrices with $a_{11} = 0$ is

$$N_0 = N - N_1 = (2^{n(n-1)/2}(2^n - 1) - 2^{(n-1)(n+2)/2}) \prod_{i=1}^{n-1}(2^i - 1) =$$

$$(2^{n-1} - 1)2^{n(n-1)/2} \prod_{i=1}^{n-1}(2^i - 1).$$

Thus

$$p = \frac{2^{n-1} - 1}{2^n - 1}.$$

So for $n = 2, 3$ we get $p = 2/3$ and $3/7$, respectively.

M2. Let $t > 0$ and b_n be the sequence defined by the recursion

$$b_0 = 1, \quad b_n = t^{-1}(b_{n-1} + \tfrac{1}{2}b_{n-2} + ... + \tfrac{1}{n}b_0).$$

(i) Show that there exists

$$b = \lim_{n \to \infty} b_n^{1/n}$$

and compute b.

(ii) Show that there exists

$$C = \lim_{n\to\infty} \frac{b_n}{b^n}$$

and compute C.

(iii) Compute $\limsup_{n\to\infty} |b_n - Cb^n|^{1/n}$.

(iv) Do (i)-(iii) for the recursion

$$b_0 = 1, \quad b_n = t^{-1} \sum_{k=1}^{n} \frac{k^{k-1}}{k!} b_{n-k}$$

with $0 < t < 1$.

(v) Compute b for this recursion if $t \geq 1$.

Solution. (i)-(iii) Consider the generating function

$$f(z) := \sum_{n=0}^{\infty} b_m z^m.$$

Then the recursion implies that

$$f(z)(1 + t^{-1}\log(1-z)) = 1.$$

Thus

$$f(z) = \frac{1}{1 + t^{-1}\log(1-z)}.$$

This function has a simple pole at the solution of the equation

$$\log(1-z) = -t$$

which gives

$$z = 1 - e^{-t}.$$

Thus

$$b = \frac{1}{1-e^{-t}}.$$

To find C, we need to compute the residue of f at its pole, i.e.,

$$C = -b \lim_{z\to b^{-1}} \frac{z - b^{-1}}{1 + t^{-1}\log(1-z)}.$$

Using L'Hospital's rule, we get

$$C = -b\frac{b^{-1}-1}{t^{-1}} = t(b-1) = \frac{te^{-t}}{1-e^{-t}} = \frac{t}{e^t - 1}.$$

The function

$$f_*(z) = f(z) - \frac{C}{1-bz}$$

has no poles for $|z| < 1$ but branches at 1. So we get

$$\limsup |b_n - Cb^n|^{1/n} = 1.$$

(iv) The analysis is similar except

$$f(z) = \frac{1}{1 - t^{-1}h(z)},$$

where

$$h(z) = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} z^n = -W(-z),$$

where $W(z)$ is the **Lambert function.** So the pole of f is at the solution z of the equation

$$h(z) = t.$$

By the Lagrange inversion theorem, the inverse to $w = h(z)$ is the function $z = we^{-w}$, so we get

$$z = te^{-t}.$$

Thus

$$b = t^{-1}e^t.$$

Also $\frac{dz}{dw} = (1-w)e^{-w}$, so $\frac{dw}{dz} = (1-w)^{-1}e^w$, and using the L'Hopital's rule, we get

$$C = \frac{b}{t^{-1}h'(b^{-1})} = \frac{tb}{(1-t)^{-1}e^t} = 1 - t.$$

Since the minimal value of the function $t^{-1}e^t$ for $t > 0$ is e (at $t = 1$), we get

$$\limsup |b_n - Cb^n|^{1/n} = e.$$

(v) For $t \geq 1$ the function $f(z)$ is analytic for $|z| < e^{-1}$ but has a singularity at $z = e^{-1}$, so $b = e$.

M3. Let S_n be the symmetric group on n elements (we agree that $S_0 = 1$). Compute $h(z) := \sum_{n=0}^{\infty} a_n z^n$, where a_n is the number of conjugacy classes (under S_n) of homomorphisms of $\phi : G \to S_n$, where

(i) $G = \mathbb{Z}/2 \times \mathbb{Z}/2$;

(ii) $G = S_3$.

(iii) Do the same for *injective* homomorphisms in (i),(ii).

(iv) How many conjugacy classes of subgroups isomorphic to S_3 are there in S_{10} ? Can you describe all of them? How many are there in S_{100}?

Solution. A homomorphism $G \to S_n$ is the same as an action of G on $[1, n]$. Under such action $[1, n]$ falls into orbits, which are classified by stabilizers. What types of stabilizers do we have (up to conjugation in G)?

(i) The stabilizers are $1, G$, and three versions of $\mathbb{Z}/2$. So the number of conjugacy classes is the number of solutions of the equation

$$n_1 + 2(n_2 + n_3 + n_4) + 4n_5 = n.$$

Thus

$$h_1(z) = \frac{1}{(1-z)(1-z^2)^3(1-z^4)}.$$

(ii) The stabilizers are $1, \mathbb{Z}/2, \mathbb{Z}/3, G$. So

$$h_2(z) = \frac{1}{(1-z)(1-z^2)(1-z^3)(1-z^6)}.$$

(iii) It's easier to count non-injective homomorphisms. For the group $G = \mathbb{Z}/2 \times \mathbb{Z}/2$, they factor through one of the $\mathbb{Z}/2$ or are trivial. So we get

$$h_1^*(z) = \frac{1}{(1-z)(1-z^2)^3(1-z^4)} - \frac{3}{(1-z)(1-z^2)} + \frac{2}{1-z}.$$

For $G = S_3$ they factor through $\mathbb{Z}/2$ or are trivial, so

$$h_2^*(z) = \frac{1}{(1-z)(1-z^2)(1-z^3)(1-z^6)} - \frac{1}{(1-z)(1-z^2)}.$$

(iv) Since every automorphism of S_3 is inner, it is enough to take the corresponding coefficient in $h_2^*(z)$ which is easy to do by hand or computer. For S_{10} one gets 12 and for 100 one gets 5457.

M4. Cubicles in a software company are arranged in n adjacent rows, 3 cubicles in each or, equivalently, 3 columns with n cubicles in each (so they look like an n by 3 chessboard). Two cubicles are adjacent if they share at least one corner. Covid social distancing protocol prohibits placing people in adjacent cubicles. If the company has k employees, let $a(n, k)$ be the number of allowable arrangements of k cubicles to be occupied by employees. For example, $a(1, 2) = 1$, $a(2, 2) = 4$, $a(3, 4) = 1$, etc.

(i) Find $a(5, 3)$.

(ii) Compute the generating function

$$A(x, y) := \sum_{k,n \geq 0} a(n, k) x^n y^k.$$

(iii) Find $a(10, 5)$.

(iv) What is the largest number of employees you can seat and how many ways to do so are there[2] (for each n)?

[2]Here, if two employees switch cubicles with each other, this counts as the same way of seating.

Solution. This is equivalent to computing the number of arrangements of unlabeled non-attacking kings. Denote by $b(n,k)$ the number of arrangements when the first and the second cubicle in the last row are eliminated. Assume $n \geq 2, k \geq 2$. So we have

$$b(n+1,k) = b(n,k-1) + a(n,k).$$

The difference $a(n,k) - a(n-1,k)$ is the number of arrangements so that at least one employee is seated in the n-th row. This can happen in several ways:

1) This employee sits in the middle of the row. The number of such arrangements is $a(n-2,k-1)$.

2) Two employees sit in the leftmost and rightmost cubicles of the n-th row. This gives $a(n-2,k-2)$.

3) One employee sits in the leftmost or rightmost position in the last row, and there is nobody else in that row. This yields $2b(n-1,k-1)$.

Thus we get

$$a(n,k) - a(n-1,k) = a(n-2,k-1) + a(n-2,k-2) + 2b(n-1,k-1).$$

Substituting, we get

$$b(n+1,k) - b(n,k-1) - b(n,k) + b(n-1,k-1) =$$
$$b(n-1,k-1) - b(n-2,k-2) + b(n-1,k-2) - b(n-2,k-3) + 2b(n-1,k-1).$$

Simplifying, we get

$$b(n+1,k) =$$
$$b(n,k-1) + b(n,k) + 2b(n-1,k-1) - b(n-2,k-2) + b(n-1,k-2) - b(n-2,k-3).$$

So setting

$$B(x,y) := \sum_{n,k \geq 0} b(n,k) x^n y^k,$$

we get that

$$B(x,y)(1 - x - xy - 2x^2y - x^2y^2 + x^3y^2 + x^3y^3) = 1,$$

i.e.,

$$B(x,y) = \frac{1}{1 - x - xy - 2x^2y - x^2y^2 + x^3y^2 + x^3y^3}.$$

So we get

$$A(x,y) = x^{-1}((1-xy)f-1) = \frac{1 + 2xy + xy^2 - x^2y^2 - x^2y^3}{1 - x - xy - 2x^2y - x^2y^2 + x^3y^2 + x^3y^3}.$$

Expanding this function (using a computer), we find

$$a(5,3) = 105, \ a(10,5) = 12438.$$

We also see that the largest number of employees that can be seated is n for even n and $n+1$ for odd n. In the odd case there is only one way to put them (at squares whose both coordinates are odd), while in the even case there are $(\frac{n}{2}+1)^2$ (all are seated in the first and third column, and there are $n/2+1$ arrangements in each).

M5. An MIT class meets 31 times. In each meeting the professor divides the students into working groups of 5 so that every two students are in the same group exactly once.

(i) How many students are there in the class?

(ii) How to make an arrangement as in (i)?

(iii) Another MIT class has 72 students. Each week, the professor divides the students into working groups of exactly 8 people on some weeks or exactly 9 people on other weeks, so that every two students are in the same group exactly once. How many weeks will the class meet?

(iv) How to make an arrangement as in (iii)?

Solution. (i) Let the number of students in the class be m. The number of pairs is then $\frac{m(m-1)}{2}$. In each meeting $2m$ pairs are implemented. So the class met $\frac{m-1}{4}$ times. Thus we get $\frac{m-1}{4}=31$, which yields $m=125$.

(ii) Label the students by vectors in the 3-dimensional space over $\mathbb{F}_5$. For each class choose a different line l through the origin in this space (there are exactly 31 such lines), and define the groups to be the lines parallel to l. It is clear that this arrangement is as required.

(iii) The number of pairs of students is $36 \cdot 71$. If there are 8 people in each group then on that week $36 \cdot 7$ pairs are formed. If there are 9 people in each group, then on that week $36 \cdot 8$ pairs are formed. So if a is the number of weeks of the first kind and b is the number of weeks of the second kind then we get $7a+8b=71$. There are two solutions, $(9,1)$ and $(1,8)$. But if there is one week with 9-student groups then the 9 students in any such group will have to be all in different groups on all other weeks, so there will have to be at least 9 groups, hence 8 students in each. Thus $(1,8)$ is impossible and the only solution is $(9,1)$, so the class lasts $9+1=10$ weeks.

(iv) Label students by points (x,y) in the plane $\mathbb{F}_9^2$ over the field $\mathbb{F}_9$ such that $y \neq 1$. Each week choose a line l through the origin and make the groups consist of points on lines parallel to l.

M6. An even number n of identical metal rods are connected into a chain of length L by hinges (so the length of each rod is L/n). The ends of the chain are pinned to a wall at points $(x_0,y_0)=(-D/2,0)$ and $(x_n,y_n)=(D/2,0)$, where $0<D<L$; otherwise the chain is hanging freely. Denote by (x_i,y_i) the coordinates of the i-th hinge,

$1 \le i \le n-1$. The potential energy of the chain is then

$$E = \sum_{i=1}^{n-1} y_i.$$

The chain settles in the equilibrium position where E is minimal.

(i) Suppose that $y_1 = -L/cn$ for some $c > 1$. Find D and (x_i, y_i) of the equilibrium position for all i.

(ii) Explain what happens when $n \to \infty$ when D is fixed.

Solution. It is clear that the equilibrium position will be symmetric. Let $a_i = \frac{n}{L}(x_i - x_{i-1})$, $b_i = \frac{n}{L}(y_i - y_{i-1})$ for $1 \le i \le n/2$, then $a_i = a_{n+1-i}$ and $b_i = -b_{n+1-i}$. We also have $a_i^2 + b_i^2 = 1$ and

$$E = \frac{L}{n} \sum_{i=1}^{n/2} (n+1-2i) b_i.$$

We need to minimize E with constraint

$$\sum_{i=1}^{n/2} a_i = \sum_{i=1}^{n/2} \sqrt{1 - b_i^2} = \frac{nD}{2L}.$$

So using Lagrange multipliers, we get the equations

$$n + 1 - 2i = -\frac{\lambda b_i}{\sqrt{1 - b_i^2}}.$$

Thus

$$(n+1-2i)^2(1 - b_i^2) = \lambda^2 b_i^2$$

which yields

$$b_i = -\frac{n+1-2i}{\sqrt{(n+1-2i)^2 + \lambda^2}}.$$

and

$$a_i = \frac{\lambda}{\sqrt{(n+1-2i)^2 + \lambda^2}}.$$

In particular,

$$-c^{-1} = b_1 = -\frac{n-1}{\sqrt{(n-1)^2 + \lambda^2}}$$

which gives

$$\lambda = (n-1)\sqrt{c^2 - 1}.$$

Thus

$$a_i = \frac{1}{\sqrt{\frac{(n+1-2i)^2}{(n-1)^2(c^2-1)} + 1}}, \quad b_i = -\frac{1}{\sqrt{\frac{(n-1)^2(c^2-1)}{(n+1-2i)^2} + 1}},$$

so

$$x_k = \frac{L}{n}\sum_{i=1}^{k}\frac{1}{\sqrt{\frac{(n+1-2i)^2}{(n-1)^2(c^2-1)}+1}}, \quad y_k = -\frac{L}{n}\sum_{i=1}^{k}\frac{1}{\sqrt{\frac{(n-1)^2(c^2-1)}{(n+1-2i)^2}+1}},$$

and

$$D = \frac{2L}{n}\sum_{i=1}^{n/2} a_i = \frac{2L}{n}\sum_{i=1}^{n/2}\frac{1}{\sqrt{\frac{(2i-1)^2}{(n-1)^2(c^2-1)}+1}}.$$

In the limit $n \to \infty$, these sums turn into integrals, and we get the catenary curve.