# How Optimal Can We Get: Stochastic and Adversarial Reinforcement Learning

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MIT

October 16, 2022

# A Cute Robot in A Cute Maze

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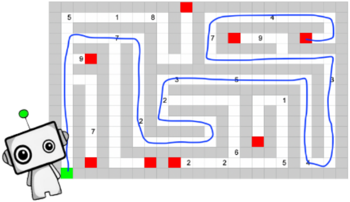

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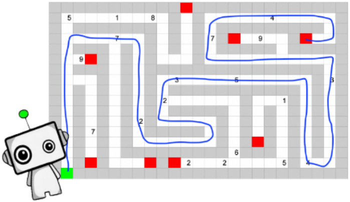

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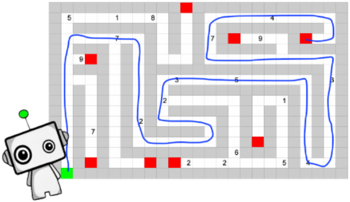

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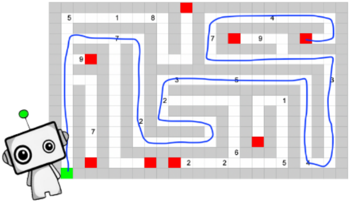

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**Let's formalize this notion...**

# Markov Decision Processes

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- $\mathcal{S}$ is **state space**: Set of all states in which the agent may be
- $\mathcal{A}$ is **action space**: Set of all actions which the agent may take in a state
- $R : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ is **reward function**: Outputs the reward given to the agent when taking action $a$ in state $s$
- $\mathcal{P} : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \to [0, 1]$ is **transition dynamics function**: Outputs the probability of the agent transitioning to new state $s'$ if it takes action $a$ in state $s$

# $\epsilon$-greedy Policy



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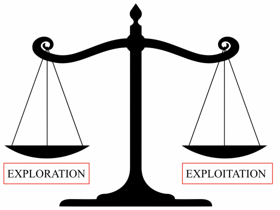

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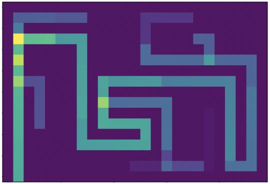
Heat map of learned Q-values:



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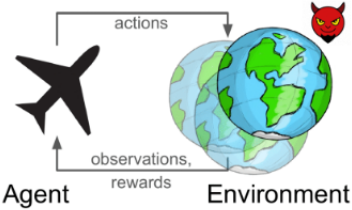
What if something perturbs the MDP?



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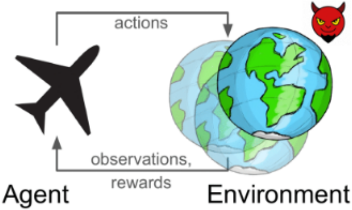


Performance can be degraded by:

- Human biases
- Modeling errors
- Actual adversaries

# Robust RL

## Definition

Robust RL aims to find the best-performing policy in the worst-case scenario. It can be framed as a 2-player zero-sum game.

Objective: Find the policy $\pi$ that satisfies:

$$\max_{\pi} \min_{\mathcal{P}} \mathbb{E}_{\pi,\mathcal{P}}\left[\sum_t R_t\right],$$

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Robust RL Methods Include:

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- Train the agent in an environment with an adversary that corrupts the reward function

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## Definition

Best of Both Worlds: We want performance that degrades gracefully with an increasing corruption level, can be used in RL

Best of Both Worlds Methods:

- Layering algorithms designed for varying corruption levels

# Problem Setting



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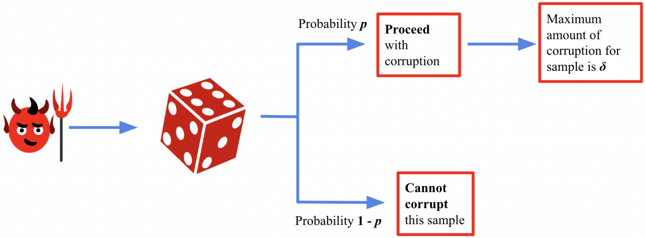

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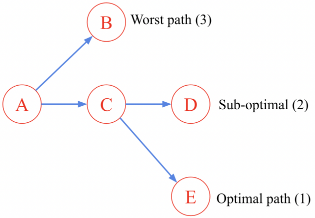

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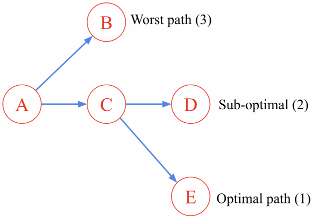


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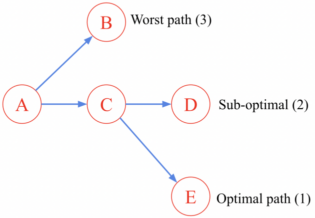


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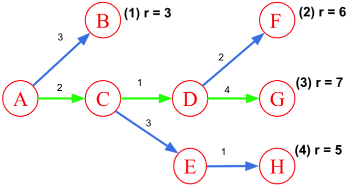
Our Approach: $2p\delta +$ extra $\frac{1}{2}p\delta$ of "free corruption" from corrupting AC whenever path 2 is traversed

# Adversarial Attack

Let's attack! Given that $p = 0.25$ and $\delta = 4; p\delta = 1$




**Algorithm 1** Adversarial Attack: Switching Non-disjoint Paths

$P \leftarrow P^*$ ▷ Current path to perturb; we will iterate and find a better one
**for all** $P_i \in \mathcal{M} \wedge P_i \neq P^*$ **do** ▷ Iterate only through suboptimal paths
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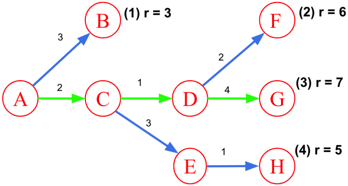


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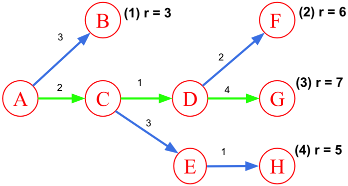


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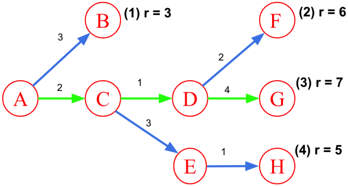


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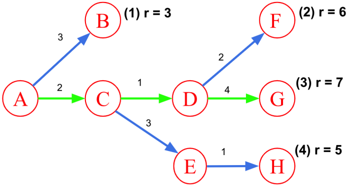


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        end for
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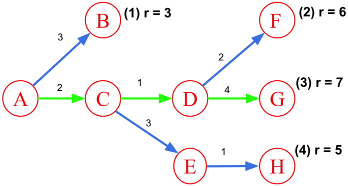


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**We choose to switch path 3 with path 4**

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   - These substitutions will not yield greater corruption since algorithm chooses edge on least number of paths, which guarantees the maximum amount.

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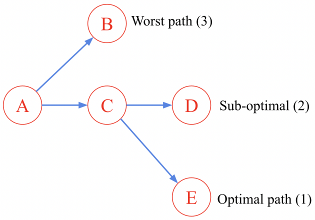

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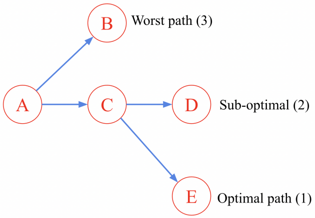


- Chebyshev's Inequality bound on expected reward of this strategy: it is less than $(r_1 + r_3) \cdot (N_1 + N_3) p\delta^2 \frac{1-p}{(r_1 - r_3)^2}$

# Future Work



- How does victim defend against adversary strategy outlined above using Best-of-Both-Worlds?
  - Devise layering algorithm for victim defense
- More generally: set up minimax between victim and adversary to fully describe their behaviors in the MDP
  - What is the value of corrupting a path that is neither the optimal path nor the path we are trying to switch with it? Is there value in confusing the victim in this way? When is this helpful?

# Acknowledgements

We would like to thank...

- MIT PRIMES; Dr. Slava Gerovitch and Dr. Srini Devadas for this wonderful opportunity
- Mayuri Sridhar for being an amazing mentor
- You!

# References



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