A Turán-type Problem in Mixed Graphs

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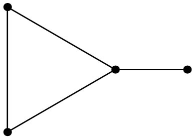
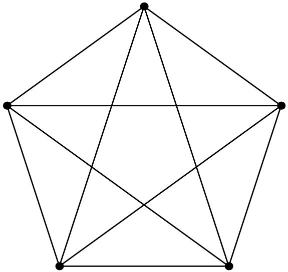
Graphs

Definition

A *graph* is a collection of vertices and edges.





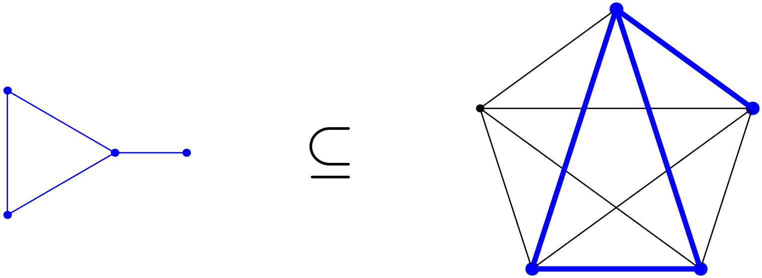
Extremal graph theory

How large/small can a graph be if it satisfies some given structural constraint?

Subgraphs

Definition

For graphs F and G, call F a *subgraph* of G (denoted $F \subseteq G$) if the vertices and edges of F are a subset of those of G.



Turán-Type Problems

For a graph with v vertices and e edges, define its *edge density* $e/\binom{v}{2}$.

Turán-type problem

Let F be a graph. What is the asymptotically maximal edge density of a graph that does not contain F as a subgraph?

Such a graph is called *F-free*.

Mantel's Theorem

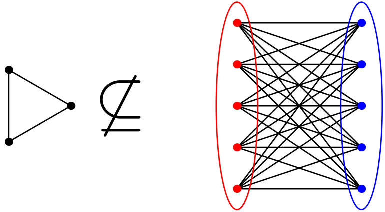
Theorem (1907, Mantel)

A triangle-free graph with n vertices contains at most $\frac{n^2}{4}$ edges.

This means the *Turán density* of the triangle is

$$\lim_{n\to\infty} \frac{n^2}{4} / \binom{n}{2} = \frac{1}{2}.$$

Example

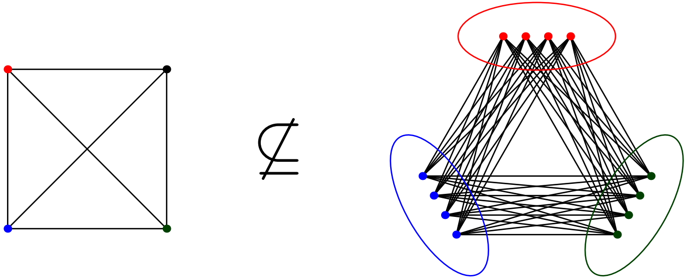


Turán's Theorem

Theorem (Turán, 1941)

The Turán density of the complete graph K_r is $\frac{r-2}{r-1}$.

Example



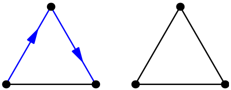
Mixed Graphs

Definition

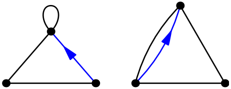
In a *mixed graph*, edges can either be *directed* or *undirected*.

Example

Mixed graphs:



Not mixed graphs:

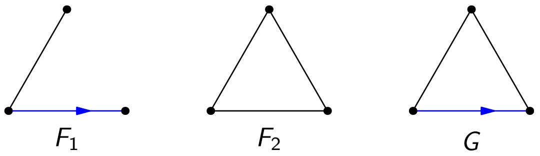


Mixed Subgraphs

Definition

F is a *subgraph* of G if F can be obtained from G by deleting vertices, deleting edges, and forgetting edge directions.

Example

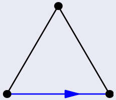


F_1 and F_2 are subgraphs of G. F_1 is not a subgraph of F_2, or vice versa.

Definition

Define the mixed graph

$$\overrightarrow{K_3} =$$



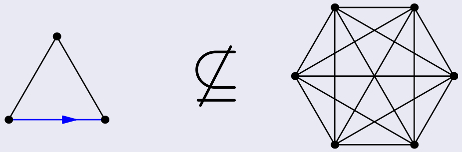
Naive Extremal Problem

Problem

What is the maximal number of edges in a $\overrightarrow{K_3}$−free graph?

Solution

As many as we can fit: $\binom{n}{2}$, where n is the number of vertices.



So perhaps this is not the question we want to ask!

Turán Density Coefficient

Definition

Let F be a mixed graph. Define the *Turán density coefficient* $\theta(F)$ as the largest value of ρ such that

$$\mathrm{undir}(G) + \rho \cdot \mathrm{dir}(G) \leq \binom{n}{2} + o(n^2)$$

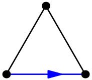
over all F-free n-vertex mixed graphs G.

This characterizes the balance between directed and undirected edges.

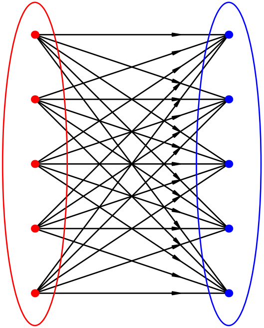
THEOREM

$$\theta(\overrightarrow{K_3}) = 2.$$

EXAMPLE



$\not\subseteq$



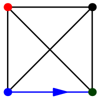
Turán's Theorem for Mixed Graphs

Theorem

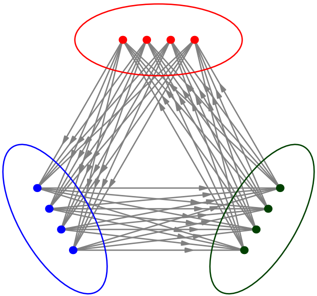
For all r,

$$\theta(\overrightarrow{K_r}) = \frac{r-1}{r-2}.$$

Example



$\not\subseteq$



Main Results

- A tight inequality for $\theta(F)$ in terms of its chromatic number: either $\theta(F) = 1, \theta(F) = \infty$, or

$$1 + \frac{1}{\chi(F)} \leq \theta(F) \leq 1 + \frac{1}{\chi(F)-2}.$$

- A variational characterization for $\theta(F)$, with "simple" asymptotically extremal graphs.
- There exists F such that $\theta(F)$ is irrational.
- $\theta(F)$ is an *algebraic number* for all F.
- For any $k \in \mathbb{N}$ there exists a family of mixed graphs F such that $\theta(F)$ has algebraic degree k.

Future Directions

- Is it possible to achieve all (or arbitrarily high) algebraic degrees with single graphs F?
- What is the set of possible values of $\theta(F)$?
- Generalize to *partially-directed hypergraphs*; applications to the k-SAT counting problem.

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