Ending States of a Special Variant of the Chip-Firing Algorithm

Rich Wang
Mentor: Tanya Khovanova

October 15, 2022
MIT PRIMES Conference

Violin Playing

- Consider a hotel with an infinite set of rooms on a number line, some of which are occupied by violinists, whom are all up late at night practicing for the big competition tomorrow.

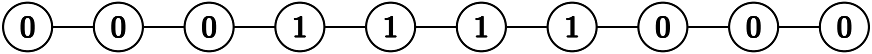
Violin Playing

- Consider a hotel with an infinite set of rooms on a number line, some of which are occupied by violinists, whom are all up late at night practicing for the big competition tomorrow.
- Two violinists in two neighboring rooms began practicing extremely loudly, annoying both of them.

Violin Playing

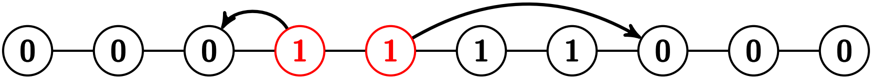
- Consider a hotel with an infinite set of rooms on a number line, some of which are occupied by violinists, whom are all up late at night practicing for the big competition tomorrow.
- Two violinists in two neighboring rooms began practicing extremely loudly, annoying both of them.
- To help them, the manager of the hotel performed the following operation: he sent one of them to the closest unoccupied room to the left, and the other to the closest unoccupied room to the right.

- Consider a hotel with an infinite set of rooms on a number line, some of which are occupied by violinists, whom are all up late at night practicing for the big competition tomorrow.
- Two violinists in two neighboring rooms began practicing extremely loudly, annoying both of them.
- To help them, the manager of the hotel performed the following operation: he sent one of them to the closest unoccupied room to the left, and the other to the closest unoccupied room to the right.

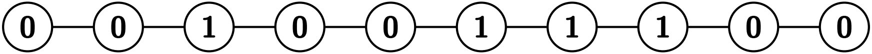


Violin Playing

- Consider a hotel with an infinite set of rooms on a number line, some of which are occupied by violinists, whom are all up late at night practicing for the big competition tomorrow.
- Two violinists in two neighboring rooms began practicing extremely loudly, annoying both of them.
- To help them, the manager of the hotel performed the following operation: he sent one of them to the closest unoccupied room to the left, and the other to the closest unoccupied room to the right.

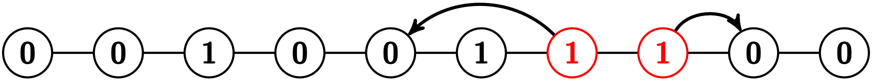


- Consider a hotel with an infinite set of rooms on a number line, some of which are occupied by violinists, whom are all up late at night practicing for the big competition tomorrow.
- Two violinists in two neighboring rooms began practicing extremely loudly, annoying both of them.
- To help them, the manager of the hotel performed the following operation: he sent one of them to the closest unoccupied room to the left, and the other to the closest unoccupied room to the right.



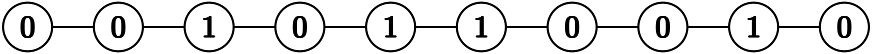
Violin Playing

- Consider a hotel with an infinite set of rooms on a number line, some of which are occupied by violinists, whom are all up late at night practicing for the big competition tomorrow.
- Two violinists in two neighboring rooms began practicing extremely loudly, annoying both of them.
- To help them, the manager of the hotel performed the following operation: he sent one of them to the closest unoccupied room to the left, and the other to the closest unoccupied room to the right.



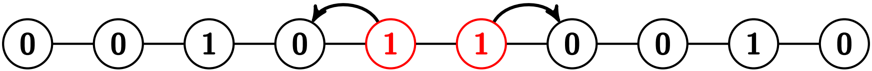
Violin Playing

- Consider a hotel with an infinite set of rooms on a number line, some of which are occupied by violinists, whom are all up late at night practicing for the big competition tomorrow.
- Two violinists in two neighboring rooms began practicing extremely loudly, annoying both of them.
- To help them, the manager of the hotel performed the following operation: he sent one of them to the closest unoccupied room to the left, and the other to the closest unoccupied room to the right.



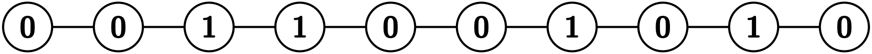
Violin Playing

- Consider a hotel with an infinite set of rooms on a number line, some of which are occupied by violinists, whom are all up late at night practicing for the big competition tomorrow.
- Two violinists in two neighboring rooms began practicing extremely loudly, annoying both of them.
- To help them, the manager of the hotel performed the following operation: he sent one of them to the closest unoccupied room to the left, and the other to the closest unoccupied room to the right.



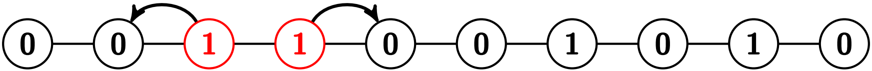
Violin Playing

- Consider a hotel with an infinite set of rooms on a number line, some of which are occupied by violinists, whom are all up late at night practicing for the big competition tomorrow.
- Two violinists in two neighboring rooms began practicing extremely loudly, annoying both of them.
- To help them, the manager of the hotel performed the following operation: he sent one of them to the closest unoccupied room to the left, and the other to the closest unoccupied room to the right.



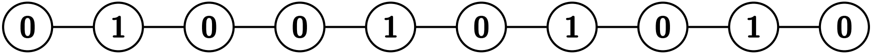
Violin Playing

- Consider a hotel with an infinite set of rooms on a number line, some of which are occupied by violinists, whom are all up late at night practicing for the big competition tomorrow.
- Two violinists in two neighboring rooms began practicing extremely loudly, annoying both of them.
- To help them, the manager of the hotel performed the following operation: he sent one of them to the closest unoccupied room to the left, and the other to the closest unoccupied room to the right.



Violin Playing

- Consider a hotel with an infinite set of rooms on a number line, some of which are occupied by violinists, whom are all up late at night practicing for the big competition tomorrow.
- Two violinists in two neighboring rooms began practicing extremely loudly, annoying both of them.
- To help them, the manager of the hotel performed the following operation: he sent one of them to the closest unoccupied room to the left, and the other to the closest unoccupied room to the right.



Making All Violinists Happy

- The operation performed resembles that of chip-firing.

Making All Violinists Happy

- The operation performed resembles that of chip-firing.
- A violinist will be happy if there are no violinists in the rooms adjacent to him.

Making All Violinists Happy

- The operation performed resembles that of chip-firing.
- A violinist will be happy if there are no violinists in the rooms adjacent to him.
- Will we always eventually reach a state in which everyone is happy?

Making All Violinists Happy

- The operation performed resembles that of chip-firing.
- A violinist will be happy if there are no violinists in the rooms adjacent to him.
- Will we always eventually reach a state in which everyone is happy?
- Yes! To prove this, we use an equivalent version of the algorithm which we call chip-pushing, which relies on the fact that we don't care about which violinist is where, only that all violinists are happy.

Making All Violinists Happy

- In chip-pushing, we take two adjacent violinists, and move the one on the right to the room directly to her right.

Making All Violinists Happy

- In chip-pushing, we take two adjacent violinists, and move the one on the right to the room directly to her right.
- If the room that she moves to is occupied, then we kick out that violinist and move her to the room directly to her right.

Making All Violinists Happy

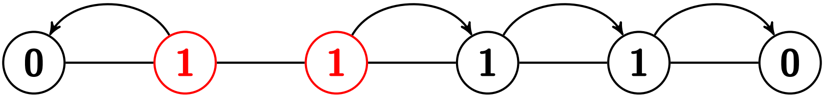
- In chip-pushing, we take two adjacent violinists, and move the one on the right to the room directly to her right.
- If the room that she moves to is occupied, then we kick out that violinist and move her to the room directly to her right.
- We repeatedly do this until a kicked out violinist finds an empty room.

Making All Violinists Happy

- In chip-pushing, we take two adjacent violinists, and move the one on the right to the room directly to her right.
- If the room that she moves to is occupied, then we kick out that violinist and move her to the room directly to her right.
- We repeatedly do this until a kicked out violinist finds an empty room.
- We then do the same with the left side.

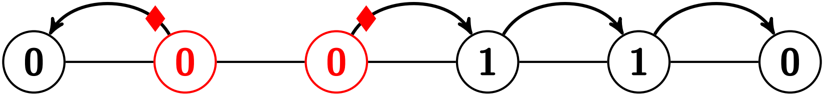
Making All Violinists Happy

- In chip-pushing, we take two adjacent violinists, and move the one on the right to the room directly to her right.
- If the room that she moves to is occupied, then we kick out that violinist and move her to the room directly to her right.
- We repeatedly do this until a kicked out violinist finds an empty room.
- We then do the same with the left side.



Making All Violinists Happy

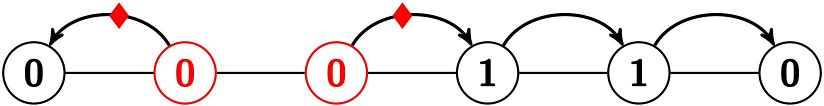
- In chip-pushing, we take two adjacent violinists, and move the one on the right to the room directly to her right.
- If the room that she moves to is occupied, then we kick out that violinist and move her to the room directly to her right.
- We repeatedly do this until a kicked out violinist finds an empty room.
- We then do the same with the left side.



- Notice that the ending state is the same after performing the move on the same pair of violinists.

Making All Violinists Happy

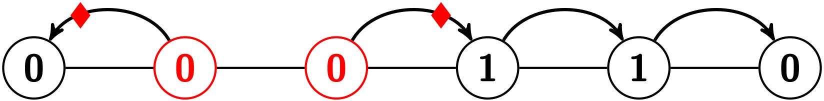
- In chip-pushing, we take two adjacent violinists, and move the one on the right to the room directly to her right.
- If the room that she moves to is occupied, then we kick out that violinist and move her to the room directly to her right.
- We repeatedly do this until a kicked out violinist finds an empty room.
- We then do the same with the left side.



- Notice that the ending state is the same after performing the move on the same pair of violinists.

Making All Violinists Happy

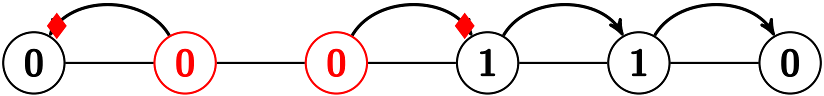
- In chip-pushing, we take two adjacent violinists, and move the one on the right to the room directly to her right.
- If the room that she moves to is occupied, then we kick out that violinist and move her to the room directly to her right.
- We repeatedly do this until a kicked out violinist finds an empty room.
- We then do the same with the left side.



- Notice that the ending state is the same after performing the move on the same pair of violinists.

Making All Violinists Happy

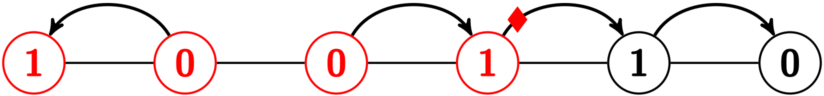
- In chip-pushing, we take two adjacent violinists, and move the one on the right to the room directly to her right.
- If the room that she moves to is occupied, then we kick out that violinist and move her to the room directly to her right.
- We repeatedly do this until a kicked out violinist finds an empty room.
- We then do the same with the left side.



- Notice that the ending state is the same after performing the move on the same pair of violinists.

Making All Violinists Happy

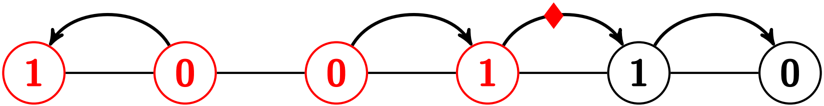
- In chip-pushing, we take two adjacent violinists, and move the one on the right to the room directly to her right.
- If the room that she moves to is occupied, then we kick out that violinist and move her to the room directly to her right.
- We repeatedly do this until a kicked out violinist finds an empty room.
- We then do the same with the left side.



- Notice that the ending state is the same after performing the move on the same pair of violinists.

Making All Violinists Happy

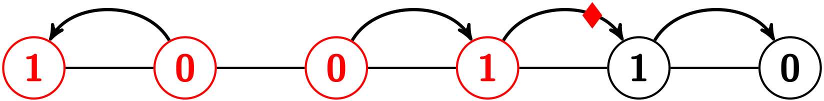
- In chip-pushing, we take two adjacent violinists, and move the one on the right to the room directly to her right.
- If the room that she moves to is occupied, then we kick out that violinist and move her to the room directly to her right.
- We repeatedly do this until a kicked out violinist finds an empty room.
- We then do the same with the left side.



- Notice that the ending state is the same after performing the move on the same pair of violinists.

Making All Violinists Happy

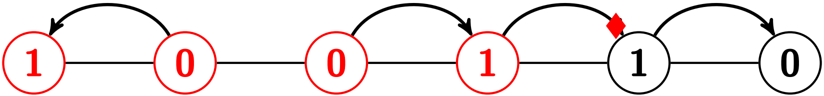
- In chip-pushing, we take two adjacent violinists, and move the one on the right to the room directly to her right.
- If the room that she moves to is occupied, then we kick out that violinist and move her to the room directly to her right.
- We repeatedly do this until a kicked out violinist finds an empty room.
- We then do the same with the left side.



- Notice that the ending state is the same after performing the move on the same pair of violinists.

Making All Violinists Happy

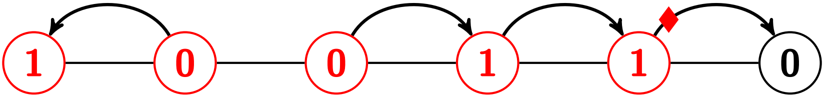
- In chip-pushing, we take two adjacent violinists, and move the one on the right to the room directly to her right.
- If the room that she moves to is occupied, then we kick out that violinist and move her to the room directly to her right.
- We repeatedly do this until a kicked out violinist finds an empty room.
- We then do the same with the left side.



- Notice that the ending state is the same after performing the move on the same pair of violinists.

Making All Violinists Happy

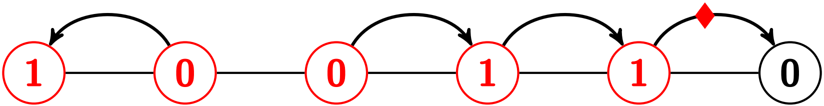
- In chip-pushing, we take two adjacent violinists, and move the one on the right to the room directly to her right.
- If the room that she moves to is occupied, then we kick out that violinist and move her to the room directly to her right.
- We repeatedly do this until a kicked out violinist finds an empty room.
- We then do the same with the left side.



- Notice that the ending state is the same after performing the move on the same pair of violinists.

Making All Violinists Happy

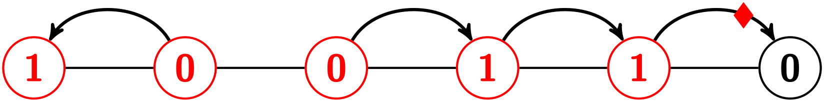
- In chip-pushing, we take two adjacent violinists, and move the one on the right to the room directly to her right.
- If the room that she moves to is occupied, then we kick out that violinist and move her to the room directly to her right.
- We repeatedly do this until a kicked out violinist finds an empty room.
- We then do the same with the left side.



- Notice that the ending state is the same after performing the move on the same pair of violinists.

Making All Violinists Happy

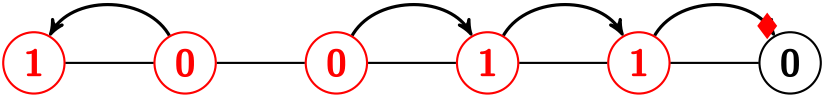
- In chip-pushing, we take two adjacent violinists, and move the one on the right to the room directly to her right.
- If the room that she moves to is occupied, then we kick out that violinist and move her to the room directly to her right.
- We repeatedly do this until a kicked out violinist finds an empty room.
- We then do the same with the left side.



- Notice that the ending state is the same after performing the move on the same pair of violinists.

Making All Violinists Happy

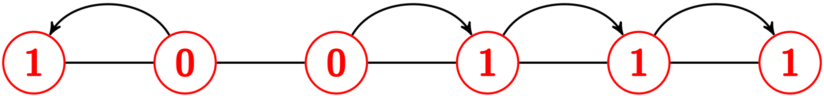
- In chip-pushing, we take two adjacent violinists, and move the one on the right to the room directly to her right.
- If the room that she moves to is occupied, then we kick out that violinist and move her to the room directly to her right.
- We repeatedly do this until a kicked out violinist finds an empty room.
- We then do the same with the left side.



- Notice that the ending state is the same after performing the move on the same pair of violinists.

Making All Violinists Happy

- In chip-pushing, we take two adjacent violinists, and move the one on the right to the room directly to her right.
- If the room that she moves to is occupied, then we kick out that violinist and move her to the room directly to her right.
- We repeatedly do this until a kicked out violinist finds an empty room.
- We then do the same with the left side.



- Notice that the ending state is the same after performing the move on the same pair of violinists.

Obtaining Universal Happiness with Chip-Pushing

- We prove that there exists an upper bound on the maximum possible number of times that the chip-pushing operation can be performed with N violinists by using induction.

Obtaining Universal Happiness with Chip-Pushing

- We prove that there exists an upper bound on the maximum possible number of times that the chip-pushing operation can be performed with N violinists by using induction.
- If we order the violinists in our starting state from left to right as v_1 through v_{N+1}, notice that the order of the violinists never changes.

Obtaining Universal Happiness with Chip-Pushing

- We prove that there exists an upper bound on the maximum possible number of times that the chip-pushing operation can be performed with N violinists by using induction.
- If we order the violinists in our starting state from left to right as v_1 through v_{N+1}, notice that the order of the violinists never changes.
- From the idea of chip-pushing, v_1 only makes moves to the left, and v_{N+1} only makes moves to the right.

Obtaining Universal Happiness with Chip-Pushing

- Assume for the sake of contradiction that we make infinite moves, and WLOG violinist v_{N+1} makes an infinite number of moves.

Obtaining Universal Happiness with Chip-Pushing

- Assume for the sake of contradiction that we make infinite moves, and WLOG violinist v_{N+1} makes an infinite number of moves.
- This tells us that at some point, we have an arbitrarily large gap between some two consecutive violinists, v_i and v_{i+1}.

Obtaining Universal Happiness with Chip-Pushing

- Assume for the sake of contradiction that we make infinite moves, and WLOG violinist v_{N+1} makes an infinite number of moves.
- This tells us that at some point, we have an arbitrarily large gap between some two consecutive violinists, v_i and v_{i+1}.
- But by our inductive assumption, the number of moves that can be performed among the i violinists v_1 through v_i is bounded, as is the number of moves that can be performed among the $N+1-i$ violinists v_{i+1} through v_{N+1}.

Obtaining Universal Happiness with Chip-Pushing

- Assume for the sake of contradiction that we make infinite moves, and WLOG violinist v_{N+1} makes an infinite number of moves.
- This tells us that at some point, we have an arbitrarily large gap between some two consecutive violinists, v_i and v_{i+1}.
- But by our inductive assumption, the number of moves that can be performed among the i violinists v_1 through v_i is bounded, as is the number of moves that can be performed among the $N+1-i$ violinists v_{i+1} through v_{N+1}.
- Because the gap between v_i and v_{i+1} is arbitrarily large, violinists from each group will never make moves with each other either. So at some point, we must be unable to make moves.

Implications of Termination

- This naturally leads us to wonder about what the ending states can be.

Implications of Termination

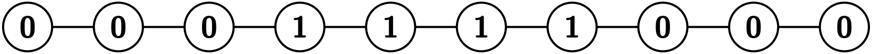
- This naturally leads us to wonder about what the ending states can be.
- We introduce an object of our study: a **clusteron**, which is an initial state that consists of a set of consecutive single-occupancy rooms.

Implications of Termination

- This naturally leads us to wonder about what the ending states can be.
- We introduce an object of our study: a **clusteron**, which is an initial state that consists of a set of consecutive single-occupancy rooms.
- We let the variable N equal the number of total violinists in the rooms of a clusteron, which we define as its **size**.

Implications of Termination

- This naturally leads us to wonder about what the ending states can be.
- We introduce an object of our study: a **clusteron**, which is an initial state that consists of a set of consecutive single-occupancy rooms.
- We let the variable N equal the number of total violinists in the rooms of a clusteron, which we define as its **size**.



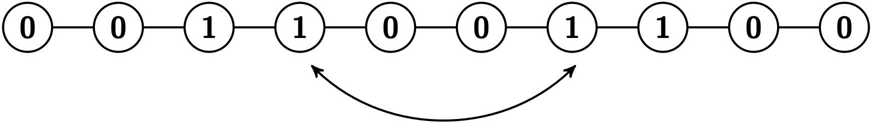
A clusteron of size 4

Ending States of Clusterons

- We define a **gap** to be the number of unoccupied rooms between two occupied rooms (that do not have any occupied rooms between them), and an **x-gap** to be a gap of size x.

Ending States of Clusterons

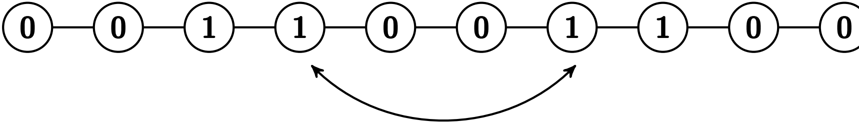
- We define a **gap** to be the number of unoccupied rooms between two occupied rooms (that do not have any occupied rooms between them), and an **x-gap** to be a gap of size x.



Example of a 2-gap

Ending States of Clusterons

- We define a **gap** to be the number of unoccupied rooms between two occupied rooms (that do not have any occupied rooms between them), and an **x-gap** to be a gap of size x.

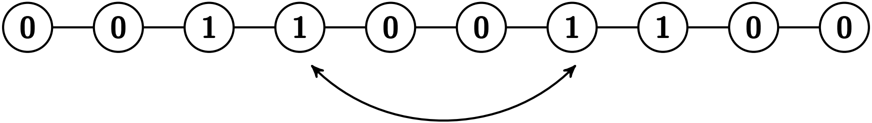


Example of a 2-gap

- Given a state S, we call the **shadow** of S to be the segment of rooms and the number of occupants in each room, starting from the first occupied room to the last occupied room.

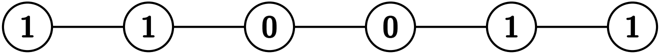
Ending States of Clusterons

- We define a **gap** to be the number of unoccupied rooms between two occupied rooms (that do not have any occupied rooms between them), and an **x-gap** to be a gap of size x.



Example of a 2-gap

- Given a state S, we call the **shadow** of S to be the segment of rooms and the number of occupants in each room, starting from the first occupied room to the last occupied room.



Shadow of the above state

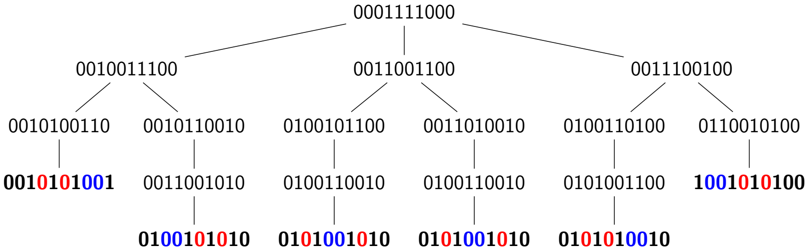
Ending States of Clusterons

Theorem (Khovanova, W. 2022)

The set of all final shadows of a clusteron, for all N, equals the set of all shadows that have 1 *two-gap and* $N-2$ *one-gaps.*

Theorem (Khovanova, W. 2022)

The set of all final shadows of a clusteron, for all N, equals the set of all shadows that have 1 two-gap and $N-2$ one-gaps.



All ending states have 1 two-gap and 2 one-gaps.

Probabilities

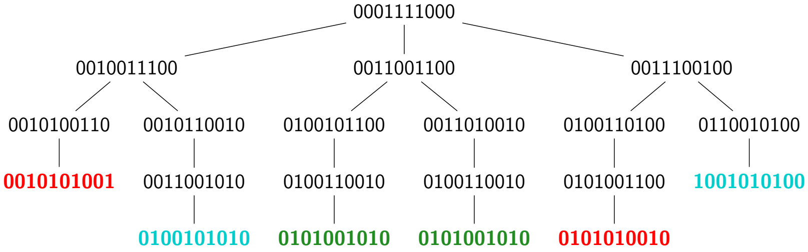
Conjecture (Khovanova, W. 2022)

In a clusteron, if at each state we uniformly select a move to perform from all possible moves, then the probability of ending with each shadow is equal.

Probabilities

Conjecture (Khovanova, W. 2022)

In a clusteron, if at each state we uniformly select a move to perform from all possible moves, then the probability of ending with each shadow is equal.



The probability of ending with each shadow is $\frac{1}{3}$.

Acknowledgements

I would like to thank

- My mentor, Dr. Tanya Khovanova for guiding me through this project, helping me with the research process, and providing her invaluable feedback on both paper and presentation.
- Dr. Darij Grinberg for suggesting the idea for this project.
- MIT PRIMES for making this research possible.

References

- Darij Grinberg, Math 235: Mathematical Problem Solving, unpublished manuscript, available at https://www.cip.ifi.lmu.de/~grinberg/t/20f/mps.pdf, 2021.
- Sam Hopkins, Thomas McConville, and James Propp, Sorting via chip-firing, math.CO arXiv:1612.06816v1.