Positivity Properties of the q-Hit Numbers

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Rook Numbers

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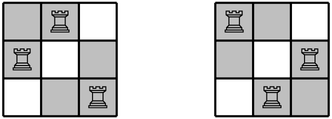

Example

In the above board B, we have $r_3(B) = 2$.

Hit Numbers

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For a board $B \subseteq [n] \times [n]$, define the hit number $h_i(B)$ as the number of ways to place n non-attacking rooks in the $[n] \times [n]$ grid such that exactly i rooks lie in B.

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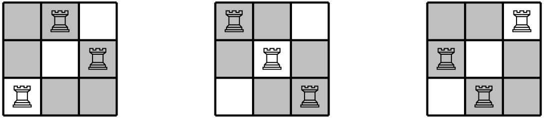

Example

In the previous board B, we have $h_2(B) = 3$.

Rook-Hit Relation

Rook-Hit Number Relation (Irving-Kaplansky, 1946)

The rook and hit numbers are related by the equation

$$\sum_{i=0}^{n} h_i(B)t^i = \sum_{i=0}^{n} r_i(B)(n-i)!(t-1)^i.$$

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Example

In the previous board, $r_3(B) = 2$, $r_2(B) = 9$, $r_1(B) = 6$, $r_0(B) = 1$, and $h_3(B) = 2$, $h_2(B) = 3$, $h_1(B) = 0$, $h_0(B) = 1$, so

$$2t^3 + 3t^2 + 0t + 1 = 2(0!)(t-1)^3 + 9(1!)(t-1)^2 + 6(2!)(t-1) + 1(3!)$$

Finite Field Matrix Counting

Definition

For a board $B \subseteq [n] \times [n]$, define $\mathfrak{m}_i(B, q)$ as the number of matrices in $\mathbb{F}_q$ (finite field of size q) with support (set of nonzero entries) in B and rank i.

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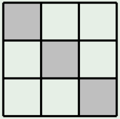

$$\mathfrak{m}_3(B) = \#\left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} : a, b, c \in \mathbb{F}_q \setminus \{0\} \right\}$$

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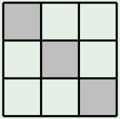

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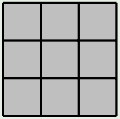

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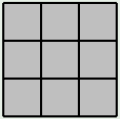

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$$= (q^3-1)(q^3-q)(q^3-q^2)$$

q-rook Numbers

Proposition (Lewis-Liu-Morales-Panova-Sam-Zhang, 2011)

We have

$$\mathfrak{m}_i(B, q) \equiv r_i(B)(q-1)^i \pmod{(q-1)^{i+1}}.$$

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Define the q-rook number $M_i(B,q) = \mathfrak{m}_i(B,q)/(q-1)^i$.

$M_i(B,q)$ is often (surprisingly) polynomial in q. If it is, then $M_i(B,1)$ must be $r_i(B)$.

q-rook Numbers

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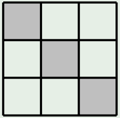
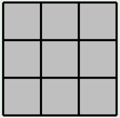


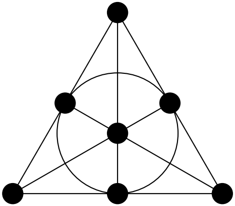
$$\mathfrak{m}_3(B,q) = (q-1)^3$$
$$M_3(B,q) = 1$$
$$M_3(B,1) = 1$$

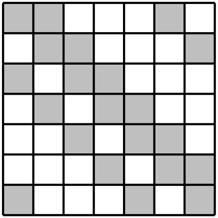
Example



$$\mathfrak{m}_3(B,q) = (q^3-1)(q^3-q)(q^3-q^2)$$
$$M_3(B,q) = q^3(q+1)(q^2+q+1)$$
$$M_3(B,1) = 6$$

Fano Plane





Example (Stembridge 1998)

$M_r(B, x+1)$ is not always a polynomial: for the Fano plane F,

$$M_7(F, x+1) = (x+1)^3(x^{11} + 17x^{10} + 135x^9 + 650x^8 + 2043x^7 + (4236 - Z_2)x^6 + 5845x^5 + 5386x^4 + 3260x^3 + 1236x^2 + 264x + 24)$$

where Z_2 is 0 if x is odd and 1 if x is even.

q-analogues

We define:

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$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}.$$
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$$[n]!_q = [n]_q[n-1]_q \ldots [1]_q.$$
$$[n]!_1 = (n)(n-1)\ldots 1 = n!.$$

q-hit Numbers

Definition (Lewis-Morales 2020)

Define the q-hit numbers $H_i(B, q)$ for a board $B \subseteq [n] \times [n]$ with the equation:

$$\sum_{i=0}^{n} H_i(B, q)t^i = q^{\binom{n}{2}} \sum_{i=0}^{n} M_i(B, q)[n-i]!_q \prod_{j=0}^{i-1}(tq^{-j} - 1).$$

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Compare to:

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We have

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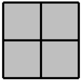

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For above board $B = [2] \times [2]$:

- $M_0(B) = 1$.
- $M_1(B) = ((q^2 - 1)(q - 1) + 2(q^2 - 1))/(q - 1) = (q + 1)^2$.
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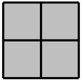

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&H_0(B, q) + H_1(B, q)t + H_2(B, q)t^2 \\
&= q(1[2 - 0]!_q + (q + 1)^2[2 - 1]!_q(t - 1) \\
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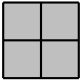

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So $H_2(B, q) = q^2 + q$. When $q = 1$, then $H_2(B, 1) = 2 = h_2(B)$.

Our Work

Conjecture

For a board B, if some polynomial P of degree $k-1$ exists where $P(x) = H_i(B, x+1) \pmod{x^k}$ for all x in a particular residue class ($x \equiv a \pmod{p}$) for some a and p), then P has nonnegative coefficients in x.

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Theorem (C-Selover, 22+)

The above is true for $k = 2$.

Acknowledgements

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