Dynkin Quivers and their Representations

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Defining Quivers

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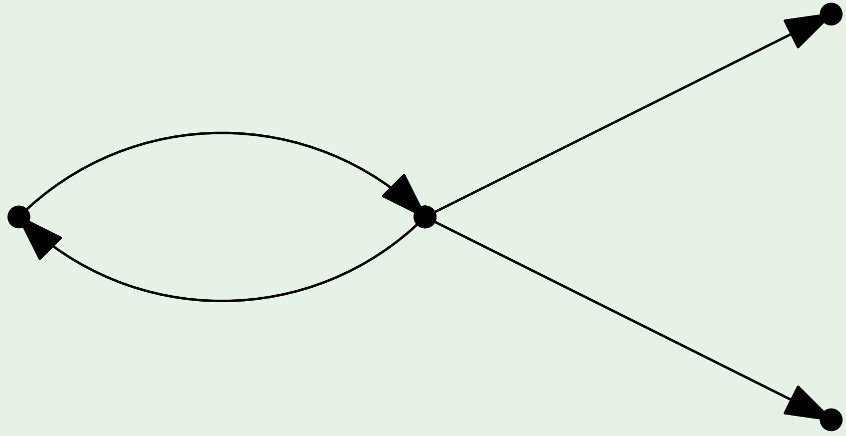

Quiver Representations

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A **representation** of a quiver is an assignment to each vertex i a vector space V_i and to each directed edge $a \to b$ a linear map $\phi_{ab} : V_a \to V_b$.

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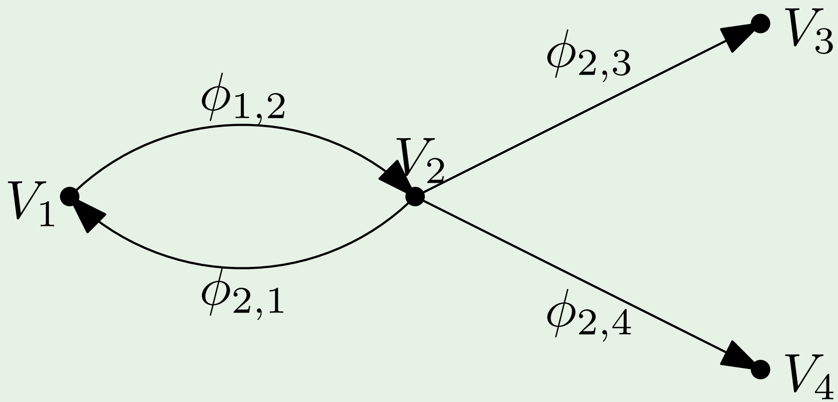

Subrepresentations

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Let (V_i, ϕ_V) be a representation of a quiver Q. Then a **subrepresentation** is a representation (W_i, ϕ_W) where $W_i \subset V_i$ for all vertices i, and $\phi_{W_{ab}}(W_a) \subset W_b$ and $\phi_{W_{ab}} = \phi_{V_{ab}} |_{W_a} \colon W_a \to W_b$ for all edges $a \to b$.

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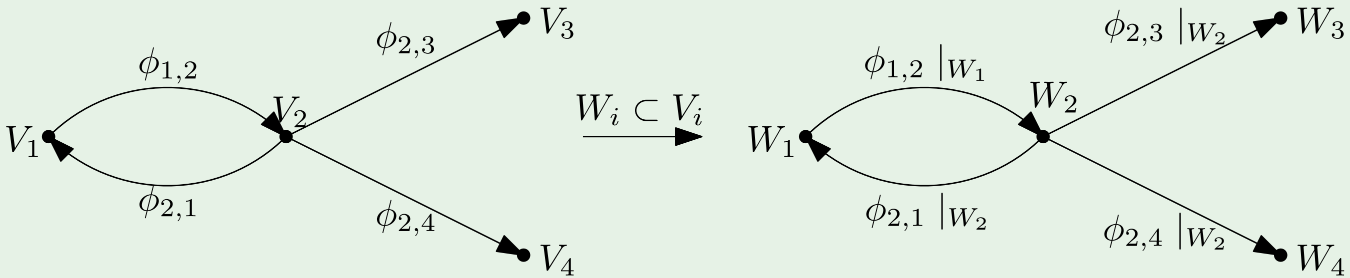

Indecomposable Representations

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Let (V_i, ϕ_V) and (Y_i, ϕ_Y) be two representations of a quiver Q. Their **direct sum** is the representation $(V_i \oplus Y_i, \phi_V \oplus \phi_Y)$.

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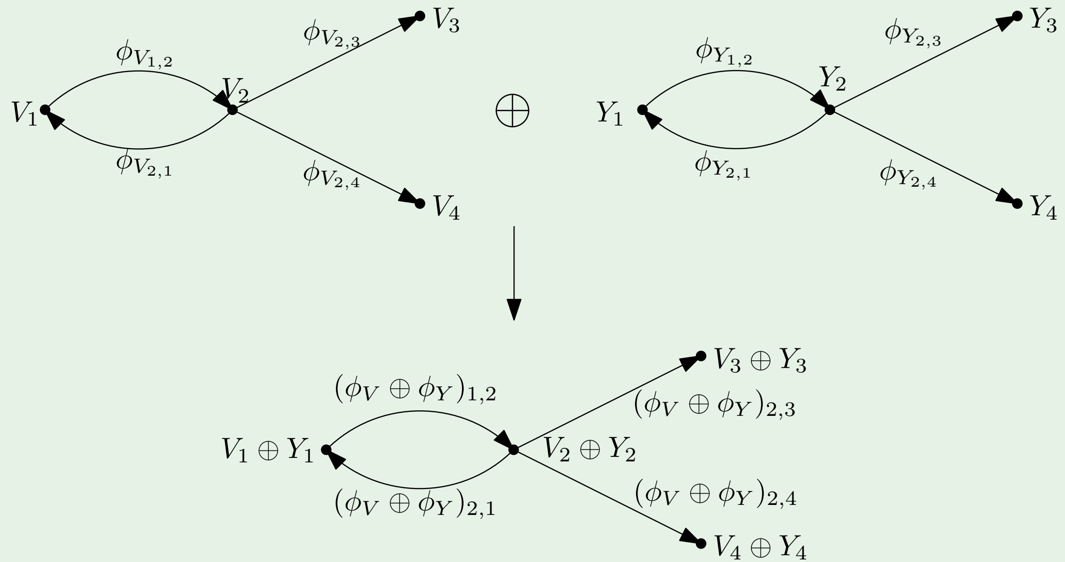

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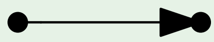

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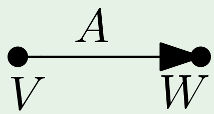

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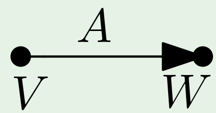

1. Let V' be the complement to the kernel of A in V, W' be the complement to the image of A in W
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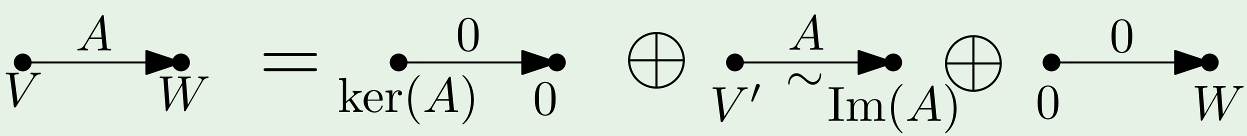

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$$V \xrightarrow{A} W$$

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2. We can decompose the representation:

$$\left(V \xrightarrow{A} W\right) = \left(\ker(A) \xrightarrow{0} 0\right) \bigoplus \left(V' \xrightarrow[\sim]{A} \mathrm{Im}(A)\right) \bigoplus \left(0 \xrightarrow{0} W'\right)$$

These are not necessarily indecomposable. Rather, they are 'multiples' of:

$$1 \longrightarrow 0 \qquad 1 \longrightarrow 1 \qquad 0 \longrightarrow 1$$

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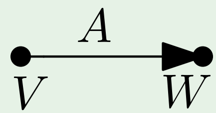

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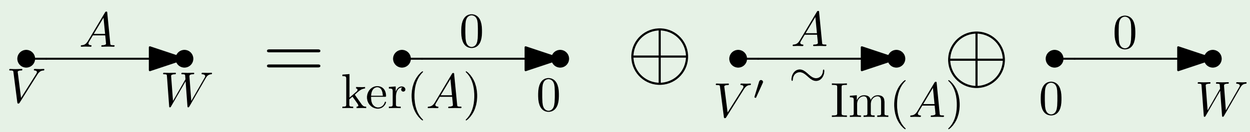

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These are the three indecomposable representations of A_2.

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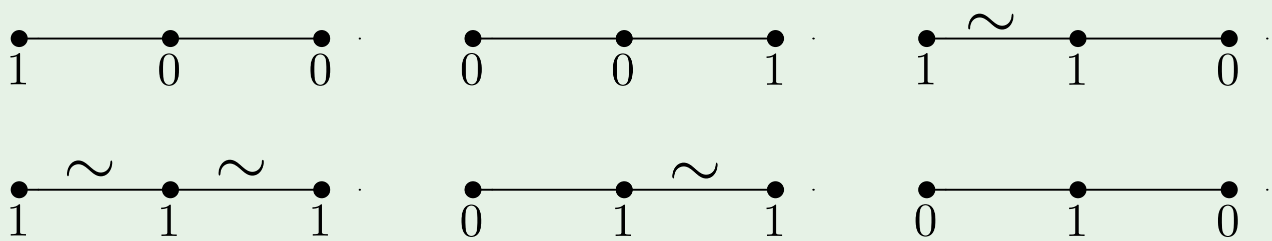
We can also write down the indecomposable representations of A_3:



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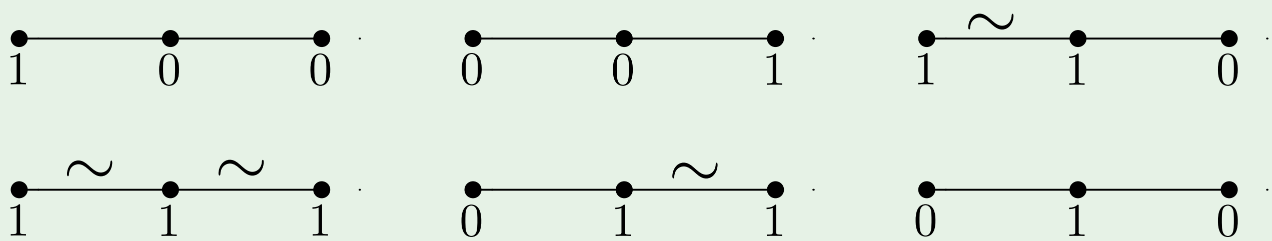

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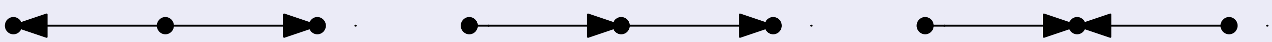

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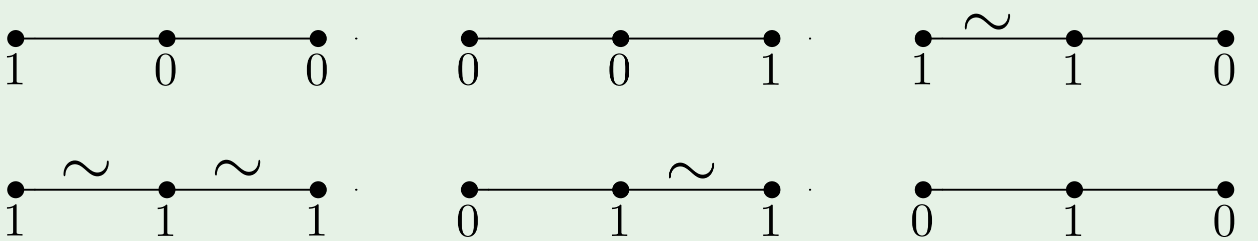

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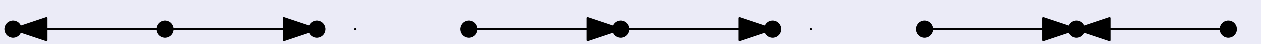

Question

- When does a quiver have finitely many indecomposable representations?

Adjacency matrix

Definition

For any quiver whose vertices are labeled $1, \ldots, n$, define the matrix R_Γ to be the *adjacency matrix* of the underlying (undirected) graph Γ. This is the matrix with entries r_{ij}, where r_{ij} is the number of edges between vertices i and j.

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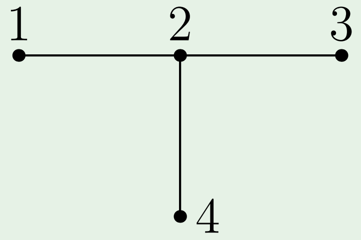

$$R_\Gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

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With the adjacency matrix R_Γ, we define the *Cartan matrix* of Γ by

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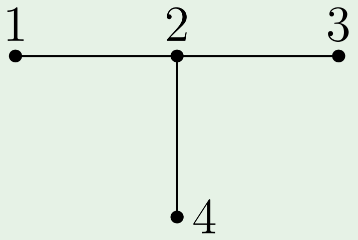

$$A_\Gamma = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$$

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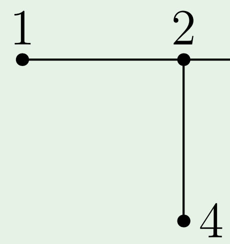

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Remark

Note that the adjacency matrix (and hence the Cartan matrix) is always symmetric.

Inner Product

For a graph Γ with n vertices and its Cartan matrix A_Γ, we define an inner product B on $\mathbb{R}^n$ by

$$B(x, y) = x^T A_\Gamma y.$$

In other words, we have

$$B(e_i, e_j) = a_{ij}$$

for basis vectors e_i, e_j, where a_{ij} is the element in the ith row and jth column in A_Γ. This is then extended linearly.

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Theorem (Gabriel)

A quiver with underlying graph Γ has finitely many indecomposable representation if and only if B is positive definite, i.e., $B(x, x) > 0$ for all $x \neq 0$.

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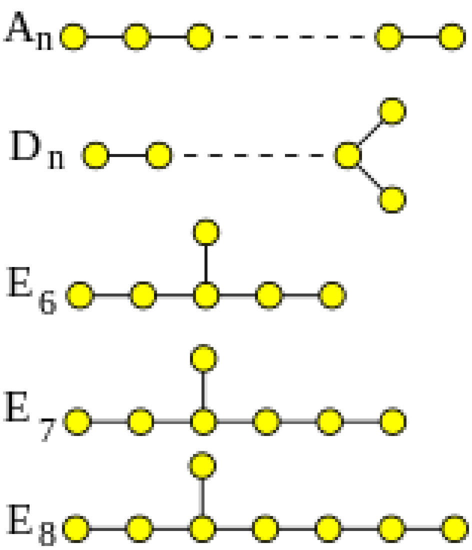
With this definition, we can fully classify all the Dynkin quivers, which are Dynkin diagrams with edges oriented.



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Note that since all entries of A_Γ are integers, we can restrict this inner product to the lattice $\mathbb{Z}^n$.

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Definition

A **root** is a nonzero vector of shortest length (with respect to the inner product) in $\mathbb{Z}^n$. For the inner product B, a root is a nonzero vector $x \in \mathbb{Z}^n$ with $B(x,x) = 2$.

Simple Roots

Remark

There are finitely many roots, since they are integer lattice points contained in some ball.

Definition

We call roots of the form

$$\alpha_i = (0, \ldots, \underbrace{1}_{i\text{th}}, \ldots, 0)$$

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Clearly these have norm $\sqrt{2}$ and span our lattice $\mathbb{Z}^n$.

Positive and Negative Roots

The choice of roots as "simple" on the previous slide is particularly good for the following reason:

Lemma

Let α be a root, and write it as a linear combination of simple roots $\alpha = \sum_{i=1}^{n} k_i \alpha_i$. Then either $k_i \geq 0$ for all i or $k_i \leq 0$ for all i.

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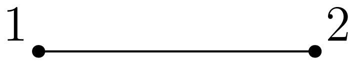
Definition

If $k_i \geq 0$ for all i, we call α a **positive root**; if $k_i \leq 0$ for all i, we call α a **negative root**.

Roots of A_2

The Cartan matrix for A_2 is given by



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$$B((x_1, x_2), (y_1, y_2)) = 2x_1y_1 + 2x_2y_2 - x_1y_2 - x_2y_1.$$

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Hence,

$$B(x, x) = 2x_1^2 + 2x_2^2 - 2x_1x_2,$$

so we can check that the only roots (when $B(x, x) = 2$) are

$$(1, 0), \qquad (0, 1), \qquad (1, 1),$$

$$(-1, 0), \qquad (0, -1), \qquad (-1, -1);$$

the first row is the positive roots while the second row is the negative roots.

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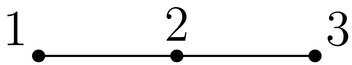

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$$B(x, x) = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3.$$

The positive roots are

$$(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (0, 1, 1), (1, 1, 1)$$

(and the negative roots are their negations).

Gabriel's theorem

Let Q be a quiver whose vertices are labeled $1, \ldots, n$. Let $V = (V_1, \ldots, V_n)$ be a representation of Q. The *dimension vector* of this representation is

$$d(V) = (\dim V_1, \ldots, \dim V_n).$$

Theorem (Gabriel)

Let Q be a Dynkin quiver. Then the dimension vector of any indecomposable representation is a positive root with respect to B. Further, for any positive root α there is exactly one indecomposable representation with dimension vector α.

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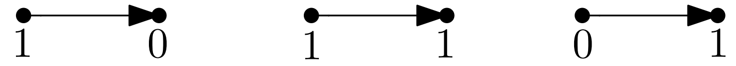

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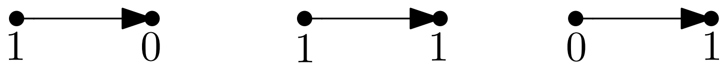

- And here are the three positive roots for A_2:

$$(1,0),(0,1),(1,1).$$

- Notice how these sets match up!

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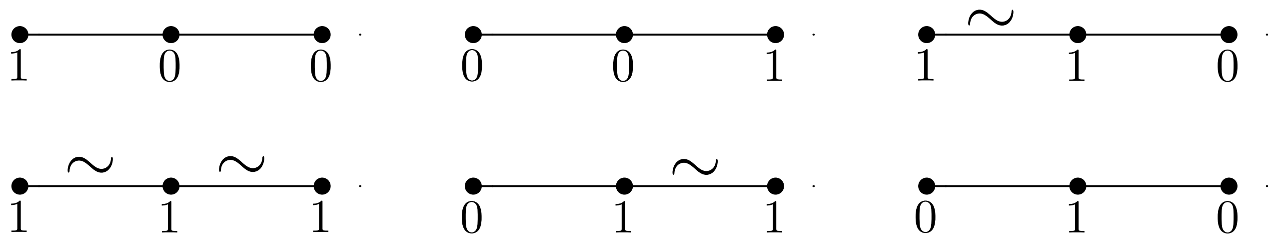

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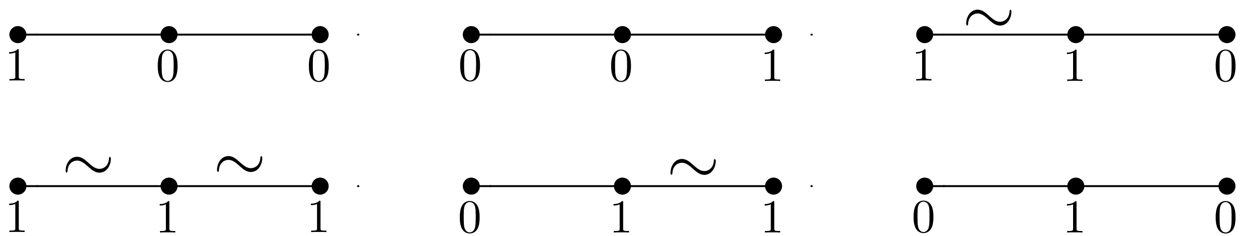

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Acknowledgements

We would like to thank

- Sasha Utiralova
- MIT PRIMES

Image credits

- Tomruen, Wikimedia Commons, https://commons.wikimedia.org/wiki/File:Simply_Laced_Dynkin_Diagrams.svg

References

Pavel Etingof.
Introduction to representation theory.
American Mathematical Society, 2011.