Game Theory

Zoe Awa and Ankita Varigonda

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Definition: Combinatorial Games

Combinatorial Games are two player games that have a set of:

- positions players can move to
- move rules for a position
- winning positions which are terminal positions

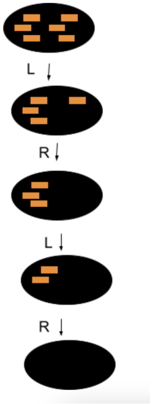
Throughout, the two players will be referred to as Richard and Louise.

Example

Pick-Up-Bricks is a game in which there is a pile of n bricks. When it is their turn, each player has the option of removing either one or two bricks from the pile. The player to take out the last brick is the winner.

Pick-Up-Bricks

The following is a sample game of Pick-Up-Bricks:



Winning Strategies

A player has a winning strategy when they possess a set of moves that guarantees them a win.

In the game Pick-Up-Bricks, there are two cases to consider:

- Case 1: When n is divisible by 3, the 2nd player has winning strategy.
- Case 2: When n is not divisible by 3, the first player has a winning strategy.

Normal Play Games

Normal Play Games consist of a set of positions, with a rule that dictates for each position, where Louise and Richard can move too. The winner of these type of games is the last player to make a move.

There are four types of positions:

- Previous Player(P): The second player to move has a winning strategy
- Next Player(N): The first player has a winning strategy
- Louise(L): Louise has a winning strategy no matter who goes first
- Richard(R): Richard has a winning strategy no matter who goes first

Positions

Normal play games can be written in position notation:

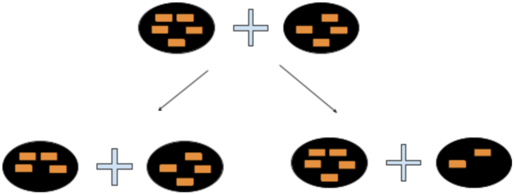
$$\gamma = \{\alpha_1 ... \alpha_n | \beta_1 ... \beta_n\}$$

The variable γ represents the game as a whole. $\alpha_1 ... \alpha_n$ represents the moves Louise can make, while $\beta_1 ... \beta_n$ represents Richard's moves.

There are two kinds of Normal play games, partizan and impartial games. Impartial games are when the two players have the same set of moves and partizan is when they do not.

Summing Games

If α and β are positions in a normal play game, then $\alpha + \beta$ is a new position made up of the elements α and β. On each turn, a player can move in either element α or β.



Equivalence

Definition

Two games a and b are equivalent(they can be different games) if and only if for every position c, the positions $a + c$ and $b + c$ have the same type.

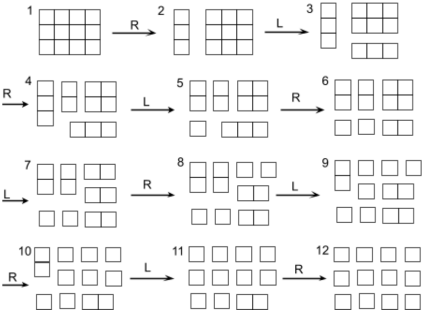
An example of this is that Pick-Up-Bricks games with 3 and 6 blocks are equivalent.

Even if two positions are from different games, if they behave the same, they are equivalent.

Cut-Cake

Cut-Cake

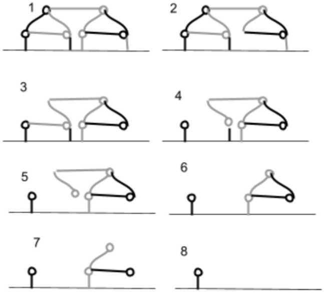
Cut-Cake is a Normal Play Game where each game position consists of uncut pieces of cake. Louise makes horizontal cuts while Richard makes vertical cuts. The game is won by the last player to make a cut.



Hackenbush

Hackenbush

Hackenbush is a game where a particular position is a graph made up of edges which are assigned one of two colors. On her turn, Louise may erase any edge which is of her color and on his turn, Richard may do the same. This graph is attached to the "ground", and after a move, all parts of the graph disconnected from the ground are removed.



Types of Hackenbush Games

Proposition 1

We will assign any Hackenbush game that has a winning strategy for the previous player with •0.

Hackenbush games such as the ones below are opposites:

Integer Games

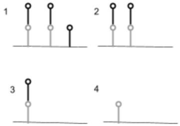
A game made of a singular grey edge, Louise's color, can be assigned the number •1:

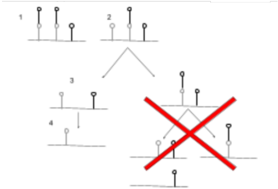
Games can also be added together:

Game

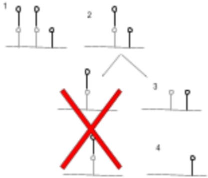
Let's analyze the game below:

If Richard goes first:

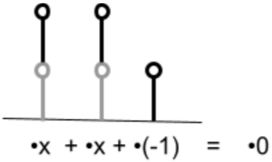




If Louise goes first:



Calculation



$$\bullet x + \bullet x + \bullet(-1) = \bullet 0$$

$$\bullet x + \bullet x = \bullet 1$$

$$x = \frac{1}{2}$$

Lemma

For every positive integer k, $\bullet\frac{1}{2^k} + \bullet\frac{1}{2^k} \equiv \bullet\frac{1}{2^{k-1}}$

Dyadic Numbers

Definition

Numbers in the form of a fraction where the denominator is a power of two.
Examples: $\frac{3}{4}, \frac{33}{128}, \frac{45}{64}$

Proposition 2

Dyadic numbers each have their own binary expansions made up of fractions with a denominator of a power of two.

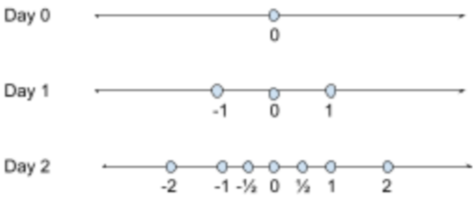
Proposition 3

Every game of Hackenbush can be represented by a dyadic number.

Birthdays

With $a_1 < a_2 < a_3 \ldots a_l$ as the numbers "born" on days $0, 1, 2 \ldots n$, the numbers born on the next day are:

1. The largest integer that is less than a_1
2. The smallest integer that is greater than a_l
3. The dyadic number $\frac{(a_i + a_{i+1})}{2}$ for $1 \leq i \leq l - 1$



Simplicity Principle

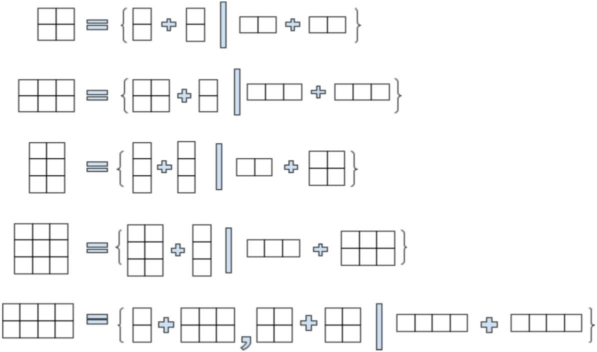
Simplicity Principle

Given a partizan game $\gamma = \{\alpha_1, \dots \alpha_m | \beta_1, \dots \beta_n\}$. Assign each position $\alpha_1, \dots \alpha_m, \beta_1, \dots \beta_n$ to Hackenbush games $\bullet\alpha_1, \dots \bullet \alpha_m, \bullet\beta_1, \dots \bullet \beta_n$ If all of $\beta_1, \dots \beta_n$ is greater than all of $\alpha_1, \dots \alpha_m$, then $\gamma \equiv \bullet c$ where c is the oldest number larger than all of $\alpha_1, \dots \alpha_m$ and smaller than all of $\beta_1, \dots \beta_n$.

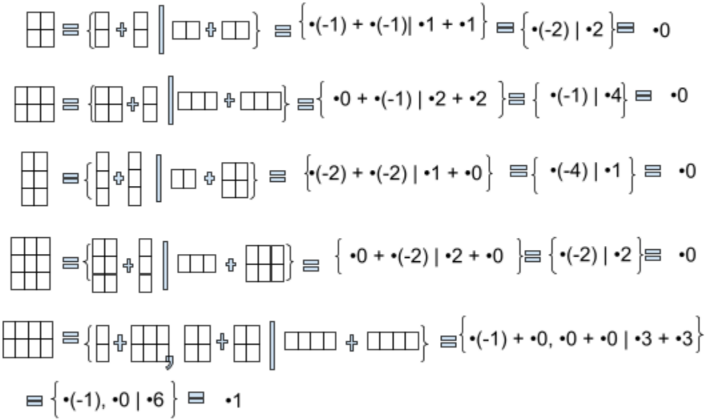
Cut-Cake

Proposition 4

A $1 \times n$ game of Cut-Cake is similar to a $\bullet(n-1)$ game of Hackenbush because both only have moves that Louise can make and she can make $n-1$ moves in each case. The same goes for an $m \times 1$ game of Cut-Cake and Richard.



Application



Thank You

Thank you for listening to our presentation. Special thanks to our mentor Yuyuan Luo and MIT Primes Circle