How to Share Your Secrets

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MIT PRIMES Computer Science Reading Group

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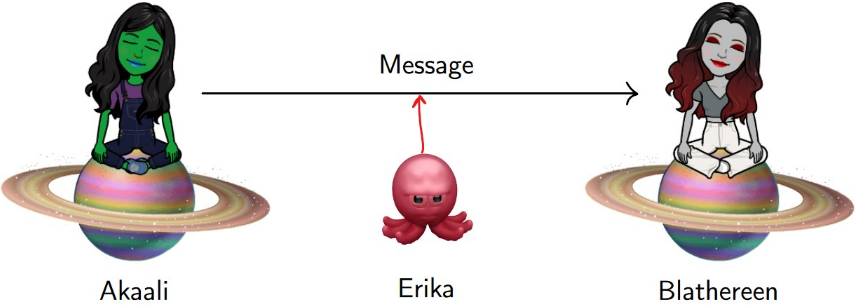
Overview

1. Secure Communication
   - Terminology
   - Defining Correctness
   - Defining Security
2. Encryption Schemes
   - One-Time Pad
   - Perfect Secrecy
3. Secret Sharing
   - Terminology
   - Correctness and Security
   - Algorithms
   - Example

Eavesdropping Erika

On the planet Osgiliath in a galaxy far, far, away...



Secure Communication

Secret-key Encryption

Components:

- Secret key, k
- Message m
- Ciphertext c

- Key Generation: $k \leftarrow Gen(1^n)$
- Encryption: $c \leftarrow Enc(k, m)$
- Decryption: $m \leftarrow Dec(k, c)$

Purpose

- Secret key k from key space $\mathcal{K}$: $k \leftarrow \mathcal{K}$
- Message m from message space $\mathcal{M}$: $m \leftarrow \mathcal{M}$
- Ciphertext c from ciphertext space $\mathcal{C}$: $c \leftarrow \mathcal{C}$

Algorithms within a cryptographic scheme:

- Key Generation Algorithm: $Gen(1^n)$: $k \leftarrow Gen$
- Encryption Algorithm: $Enc(k, m)$: $\mathcal{K} \times \mathcal{M} \rightarrow \mathcal{C}$
- Decryption Algorithm: $Dec(k, c)$: $\mathcal{K} \times \mathcal{C} \rightarrow \mathcal{M}$

Purpose: If Akaali sends over m as c, Blathereen should be able to use k to correctly determine m.

Definition of Correctness

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Consider $Enc(k, m) = m^k$ and $Dec(k, c) = \sqrt[k]{c}$.

- Let $k = 3$. Then, $Dec(k, Enc(k, m)) = \sqrt[3]{m^3} = m$.
- Let $k = 2$. Then, for $m < 0$, $Dec(k, Enc(k, m)) = \sqrt[2]{m^2} = -m$.

Definition of Security

Shannon's Perfect Secrecy

$$\forall \mathcal{M}\ \forall m \in Supp(\mathcal{M}), \forall c \in Supp(\mathcal{C}),$$
$$Pr[\mathcal{M} = m | Enc(\mathcal{K}, \mathcal{M}) = c] = Pr[\mathcal{M} = m]$$

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Theorem

An encryption scheme (Gen, Enc, Dec) satisfies perfect secrecy if and only if it satisfies perfect indistinguishability.

Encryption Schemes

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    - XOR bitwise operator: $11 \oplus 10 = 01$ (commutative)
- *Dec(k, c)*: $m = c \oplus k$

$$\begin{aligned} m &= c \oplus k \\ &= m \oplus k \oplus k \\ &= m \oplus 0^n \\ &= m. \end{aligned}$$

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- First digits of (m, k) either (1, 0) or (0, 1)
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One-Time Pad

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- First digits of (m, k) either (1, 0) or (0, 1)
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Thus, there are 2^n possibilities for (m, k).

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Two-Time Pad Attack

Consider distinct messages m_1 and m_2. Then, for the chosen key k, their ciphers are $c_1 = m_1 \oplus k$ and $c_2 = m_2 \oplus k$.

One-Time Pad

Two-Time Pad Attack

Consider distinct messages m_1 and m_2. Then, for the chosen key k, their ciphers are $c_1 = m_1 \oplus k$ and $c_2 = m_2 \oplus k$. Information leak:

$$\begin{aligned} c_1 \oplus c_2 &= (m_1 \oplus k) \oplus (m_2 \oplus k) \\ &= m_1 \oplus m_2 \oplus k \oplus k \\ &= m_1 \oplus m_2. \end{aligned}$$

Perfect Secrecy

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Proof:

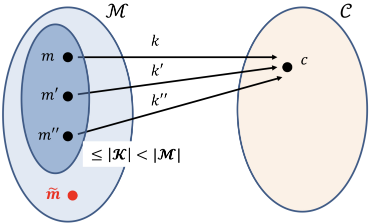

Figure 1: Prof. Vinod Vaikuntanathan's slides for 6.875 at MIT

- Every key is distinct
- One-time pad: n-bit message m; $k \xleftarrow{r} \{0,1\}^n$

Pseudorandom Generators (PRG)

Pseudorandom Generators: seed $\rightarrow b_1, b_2, b_3...$

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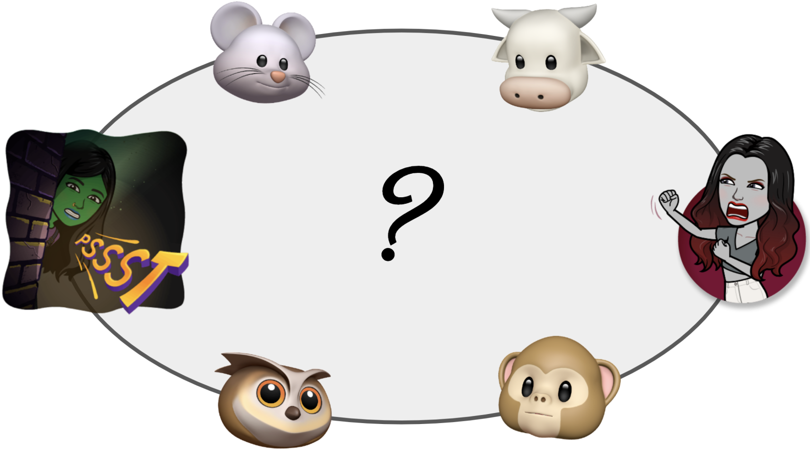
1) $m > n$, and
2) For every probabilistic polynomial time (PPT) algorithm D, there is a negligible function μ such that:

$$|\Pr[D(G(U_n)) = 0] - \Pr[D(U_m)] = 0| = \mu(n)$$

However…



How can Akaali and Blathereen make sure that the secret stays hidden?

Secret Sharing

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Goal: divide a secret into n components, where at least $1 \leq t \leq n$ components are needed to reconstruct the full secret.

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An (n, t) sharing scheme consists of:

- Share(secret s): outputs $\{s_1, s_2, ..., s_n\}$
- Reconstruct($I, \{s_i\}_{i \in I}$): outputs s if $I \subseteq \{1, 2, ..., n\}$ where $|I| \geq t$.

Notions

Correctness

For all secrets s,

- $\text{Share}(s) \rightarrow \{s_1, s_2, ..., s_n\}$
- For any $I \subseteq \{1, 2, ..., n\}$ where $|I| \geq t$,
  $\text{Reconstruct}(I, \{s_i\}_{i \in I}) \rightarrow s$.

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Security

For all $I \in \{1, 2, ..., n\}$ where $|I| < t$, $\{s_i\}_{i \in I}$ should reveal no information about s.

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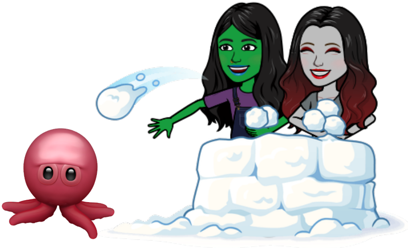
Additive Construction

- Share(s): n numbers adding up to an encoding of s
- Requires all n people to come together

Secret Sharing

Please flip over the cards we handed out, in order!



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PRIMES!

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- Secure Communication
    - Secret Key Encryption
    - Public Key Encryption
- Secret Sharing
    - Shamir's Secret Sharing Algorithm

Acknowledgements

We would like to thank...

- ...our PRIMES mentors Lalita Devadas and Alexandra Henzinger,
- ...our parents,
- ...and the PRIMES coordinators for this amazing opportunity!